

VISIBILITY AND DIRECTIONS IN QUASICRYSTALS

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ABSTRACT. It is well known that a positive proportion of all points in a d -dimensional lattice is visible from the origin, and that these visible lattice points have constant density in \mathbb{R}^d . In the present paper we prove an analogous result for a large class of quasicrystals, including the vertex set of a Penrose tiling. We furthermore establish that the statistical properties of the directions of visible points are described by certain $\mathrm{SL}(d, \mathbb{R})$ -invariant point processes. Our results imply in particular existence and continuity of the gap distribution for directions in certain two-dimensional cut-and-project sets. This answers some of the questions raised by Baake et al. in [arXiv:1402.2818].

1. INTRODUCTION

A point set $\mathcal{P} \subset \mathbb{R}^d$ has constant density in \mathbb{R}^d if there exists $\theta(\mathcal{P}) < \infty$ such that, for any bounded $\mathcal{D} \subset \mathbb{R}^d$ with boundary of Lebesgue measure zero,

$$(1.1) \quad \lim_{T \rightarrow \infty} \frac{\#(\mathcal{P} \cap T\mathcal{D})}{T^d} = \theta(\mathcal{P}) \mathrm{vol}(\mathcal{D}).$$

We refer to $\theta(\mathcal{P})$ as the density of \mathcal{P} . It is interesting to compare the density of \mathcal{P} with the density of the subset of *visible* points given by

$$(1.2) \quad \widehat{\mathcal{P}} = \{\mathbf{y} \in \mathcal{P} : t\mathbf{y} \notin \mathcal{P} \forall t \in (0, 1)\}.$$

This definition assumes that the observer is at the origin $\mathbf{0}$. Note also that, by definition, $\mathbf{0} \notin \widehat{\mathcal{P}}$. A classic example is the set of integer lattice points $\mathcal{P} = \mathbb{Z}^d$. In this case, the set of visible points is given by the primitive lattice points $\widehat{\mathcal{P}} = \{\mathbf{m} \in \mathbb{Z}^d : \gcd(\mathbf{m}) = 1\}$. Both sets have constant density with $\theta(\mathcal{P}) = 1$ and $\theta(\widehat{\mathcal{P}}) = 1/\zeta(d)$, where $\zeta(d)$ denotes the Riemann zeta function.

In this paper we are interested in the visible points of a regular cut-and-project set $\mathcal{P} = \mathcal{P}(\mathcal{W}, \mathcal{L})$ constructed from a (possibly affine) lattice $\mathcal{L} \subset \mathbb{R}^{d+m}$ and a window set $\mathcal{W} \subset \mathbb{R}^m$ (see Section 2 for detailed definitions). Such \mathcal{P} are also referred to as (Euclidean) model sets. Our first observation is the following.

Theorem 1. *If $\mathcal{P} = \mathcal{P}(\mathcal{W}, \mathcal{L})$ is a regular cut-and-project set, then \mathcal{P} and $\widehat{\mathcal{P}}$ have constant density with $0 < \theta(\widehat{\mathcal{P}}) \leq \theta(\mathcal{P})$.*

The constant density of \mathcal{P} is a well known fact [6, 21, 15]. The main point of Theorem 1 is that the *visible* set $\widehat{\mathcal{P}}$ also has a strictly positive constant density. Although for cut-and-project sets \mathcal{P} with generic choices of \mathcal{L} we have $\theta(\widehat{\mathcal{P}}) = \theta(\mathcal{P})$, there are important examples with $\theta(\widehat{\mathcal{P}}) < \theta(\mathcal{P})$. The Penrose tilings and other cut-and-project sets which are based on the construction in [10, Sec. 2.2] fall into this category, cf. [1, 16]. In some special cases, such as the Ammann-Beenker model, the visible set $\widehat{\mathcal{P}}$ can be explicitly described by a simple condition in the cut-and-project construction, see [1, Ch. 10.4] for details.

The second result of this paper concerns the distribution of directions in \mathcal{P} . Consider a general point set with constant density $\theta(\mathcal{P}) > 0$ (\mathcal{P} may be the visible set itself). We

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write $\mathcal{P}_T = \mathcal{P} \cap \mathcal{B}_T^d \setminus \{\mathbf{0}\}$ for the subset of points lying in the punctured open ball of radius T , centered at the origin. The number of such points is $\#\mathcal{P}_T \sim v_d \theta(\mathcal{P}) T^d$ as $T \rightarrow \infty$, where $v_d = \text{vol}(\mathcal{B}_1^d) = \pi^{d/2}/\Gamma(\frac{d+2}{2})$ is the volume of the unit ball. For each T , we study the directions $\|\mathbf{y}\|^{-1}\mathbf{y} \in S_1^{d-1}$ with $\mathbf{y} \in \mathcal{P}_T$, counted *with* multiplicity (if $\mathcal{P} = \widehat{\mathcal{P}}$ then the multiplicity is naturally one). The asymptotics (1.1) implies that, as $T \rightarrow \infty$, the directions become uniformly distributed on S_1^{d-1} . That is, for any set $\mathfrak{U} \subset S_1^{d-1}$ with boundary of measure zero (with respect to the volume element ω on S_1^{d-1}) we have

$$(1.3) \quad \lim_{T \rightarrow \infty} \frac{\#\{\mathbf{y} \in \mathcal{P}_T : \|\mathbf{y}\|^{-1}\mathbf{y} \in \mathfrak{U}\}}{\#\mathcal{P}_T} = \frac{\omega(\mathfrak{U})}{\omega(S_1^{d-1})}.$$

Recall that $\omega(S_1^{d-1}) = d v_d$.

To understand the fine-scale distribution of the directions in \mathcal{P}_T , we consider the probability of finding r directions in a small open disc $\mathfrak{D}_T(\sigma, \mathbf{v}) \subset S_1^{d-1}$ with random center $\mathbf{v} \in S_1^{d-1}$ and volume $\omega(\mathfrak{D}_T(\sigma, \mathbf{v})) = \frac{\sigma d}{\theta(\mathcal{P}) T^d}$ with $\sigma > 0$ fixed. Denote by

$$(1.4) \quad \mathcal{N}_T(\sigma, \mathbf{v}, \mathcal{P}) = \#\{\mathbf{y} \in \mathcal{P}_T : \|\mathbf{y}\|^{-1}\mathbf{y} \in \mathfrak{D}_T(\sigma, \mathbf{v})\}$$

the number of points in $\mathfrak{D}_T(\sigma, \mathbf{v})$. The scaling of the disc size ensures that the expectation value for the counting function is asymptotically equal to σ . That is, for any probability measure λ on S_1^{d-1} with continuous density,

$$(1.5) \quad \lim_{T \rightarrow \infty} \int_{S_1^{d-1}} \mathcal{N}_T(\sigma, \mathbf{v}, \mathcal{P}) d\lambda(\mathbf{v}) = \sigma.$$

This fact follows directly from (1.1). In the following, we denote by

$$(1.6) \quad \kappa_{\mathcal{P}} := \frac{\theta(\widehat{\mathcal{P}})}{\theta(\mathcal{P})}$$

the relative density of visible points in \mathcal{P} . We will prove:

Theorem 2. *Let $\mathcal{P} = \mathcal{P}(\mathcal{W}, \mathcal{L})$ be a regular cut-and-project set, $\sigma > 0$, $r \in \mathbb{Z}_{\geq 0}$, and let λ be a Borel probability measure on S_1^{d-1} which is absolutely continuous with respect to ω . Then the limits*

$$(1.7) \quad E(r, \sigma, \mathcal{P}) := \lim_{T \rightarrow \infty} \lambda(\{\mathbf{v} \in S_1^{d-1} : \mathcal{N}_T(\sigma, \mathbf{v}, \mathcal{P}) = r\}),$$

$$(1.8) \quad E(r, \sigma, \widehat{\mathcal{P}}) := \lim_{T \rightarrow \infty} \lambda(\{\mathbf{v} \in S_1^{d-1} : \mathcal{N}_T(\sigma, \mathbf{v}, \widehat{\mathcal{P}}) = r\})$$

exist, are continuous in σ and independent of λ . For $\sigma \rightarrow 0$ we have

$$(1.9) \quad E(0, \sigma, \mathcal{P}) = 1 - \kappa_{\mathcal{P}} \sigma + o(\sigma),$$

$$(1.10) \quad E(0, \sigma, \widehat{\mathcal{P}}) = 1 - \sigma + o(\sigma).$$

This theorem generalizes our previous work on directions in Euclidean lattices [9, Section 2]. The existence of the limit (1.7) has already been established in [10, Thm. A.1]. It is worthwhile noting that, if the set of directions in \mathcal{P} were independent and uniformly distributed random variables in S_1^{d-1} , then (1.7) would converge almost surely to the Poisson distribution

$$(1.11) \quad E(r, \sigma) = \frac{\sigma^r}{r!} e^{-\sigma}.$$

Although (1.10) is consistent with the Poisson distribution, we will see in Section 3 that $E(r, \sigma, \widehat{\mathcal{P}})$ is characterized by a certain point process in \mathbb{R}^d which is determined by a finite-dimensional probability space.

Since λ is arbitrary in Theorem 2, the result can readily be extended to cases where \mathcal{P} is exhausted by more general expanding d -dimensional domains in place of the balls \mathcal{B}_T^d . We make this precise in the appendix.

Theorem 2 allows us to answer a recent question of Baake et al. [2] on the existence of the gap distribution for the directions in the class of two-dimensional cut-and-project sets considered here. In dimension $d = 2$, it is convenient to identify the circle S_1^1 with the unit interval mod 1, and represent the set of directions in \mathcal{P}_T as $\frac{1}{2\pi} \arg(y_1 + iy_2)$ with $\mathbf{y} = (y_1, y_2) \in \mathcal{P}_T$. We label these numbers (with multiplicity) in increasing order by

$$(1.12) \quad -\frac{1}{2} < \xi_{T,1} \leq \xi_{T,2} \leq \cdots \leq \xi_{T,N(T)} \leq \frac{1}{2},$$

where $N(T) := \#\mathcal{P}_T$. The analogous construction for the visible set $\widehat{\mathcal{P}}$ yields the multiplicity-free set of directions

$$(1.13) \quad -\frac{1}{2} < \widehat{\xi}_{T,1} < \widehat{\xi}_{T,2} < \cdots < \widehat{\xi}_{T,\widehat{N}(T)} \leq \frac{1}{2}$$

where $\widehat{N}(T) := \#\widehat{\mathcal{P}}_T \leq N(T)$. We also set $\xi_{T,0} = \widehat{\xi}_{T,0} = \xi_{T,N(T)} - 1 = \widehat{\xi}_{T,\widehat{N}(T)} - 1$.

Corollary 3. *If $\mathcal{P} = \mathcal{P}(\mathcal{W}, \mathcal{L})$ is a regular cut-and-project set in dimension $d = 2$, there exists a continuous decreasing function F on $\mathbb{R}_{\geq 0}$ satisfying $F(0) = 1$ and $\lim_{s \rightarrow \infty} F(s) = 0$, such that for every $s \geq 0$,*

$$(1.14) \quad \lim_{T \rightarrow \infty} \frac{\#\{1 \leq j \leq \widehat{N}(T) : \widehat{N}(T)(\widehat{\xi}_{T,j} - \widehat{\xi}_{T,j-1}) \geq s\}}{\widehat{N}(T)} = F(s)$$

and

$$(1.15) \quad \lim_{T \rightarrow \infty} \frac{\#\{1 \leq j \leq N(T) : N(T)(\xi_{T,j} - \xi_{T,j-1}) \geq s\}}{N(T)} = \begin{cases} 1 & \text{if } s = 0 \\ \kappa_{\mathcal{P}} F(\kappa_{\mathcal{P}} s) & \text{if } s > 0. \end{cases}$$

It follows from the properties of $F(s)$ that the limit distribution function in (1.15) is continuous at $s = 0$ if and only if $\kappa_{\mathcal{P}} = 1$.

In the special case when $\mathcal{P} = \mathbb{Z}^2$, (1.14) was proved earlier by Boca, Cobeli and Zaharescu [3], who also gave an explicit formula for the limit distribution. More generally for \mathcal{P} any affine lattice in \mathbb{R}^2 , Corollary 3 was proved in [9, Thm. 1.3, Cor. 2.7].

Baake et al. [2] have observed numerically that the limiting gap distribution in Corollary 3 may vanish near zero. In Section 12 we will explain this hard-core repulsion between visible directions in the case of two-dimensional cut-and-project sets constructed over algebraic number fields, including any \mathcal{P} associated with a Penrose tiling. There is, however, no hard-core repulsion for *typical* two-dimensional cut-and-project sets. The phenomenon can be completely ruled out in higher dimensions $d \geq 3$, where we show that $E(0, \sigma, \widehat{\mathcal{P}}) > 1 - \sigma$ for all $\sigma > 0$.

The organization of this paper is as follows. In Section 2 we recall the definition of a cut-and-project set of a higher-dimensional lattice. In Section 3 we construct random point processes in \mathbb{R}^d whose realizations yield the visible points in certain $\mathrm{SL}(d, \mathbb{R})$ -invariant families of cut-and-project sets. These point processes describe the limit distributions in Theorem 2, cf. Theorem 4 in Section 3. This follows closely the construction in [10] for the full cut-and-project set. An important technical tool in our approach is the Siegel-Veech formula, which is stated and proved in Section 4. In Section 5 we describe the small- σ asymptotics of the void distribution in (1.9) and (1.10). Sections 6–9 are devoted to the proof of Theorem 1, Sections 10 and 11 to the proofs of Theorem 2 and Corollary 3, respectively. Finally in Section 12 we discuss the possible vanishing of the limiting gap distribution near zero.

2. CUT-AND-PROJECT SETS

We start by recalling the definition of a cut-and-project set in \mathbb{R}^d using our notation in [10]. These sets are also known as (Euclidean) model sets. We refer the reader to the recent monograph [1] and the surveys [13, 14] for a comprehensive introduction.

Denote by π and π_{int} the orthogonal projection of $\mathbb{R}^n = \mathbb{R}^d \times \mathbb{R}^m$ onto the first d and last m coordinates. We refer to \mathbb{R}^d and \mathbb{R}^m as the *physical space* and *internal space*, respectively. Let $\mathcal{L} \subset \mathbb{R}^n$ be a lattice of full rank. Then the closure of the set $\pi_{\mathrm{int}}(\mathcal{L})$ is an abelian subgroup \mathcal{A} of \mathbb{R}^m . We denote by \mathcal{A}° the connected subgroup of \mathcal{A} containing $\mathbf{0}$; then \mathcal{A}° is a linear

subspace of \mathbb{R}^m , say of dimension m_1 , and there exist $\mathbf{b}_1, \dots, \mathbf{b}_{m_2} \in \mathcal{L}$ ($m = m_1 + m_2$) such that $\pi_{\text{int}}(\mathbf{b}_1), \dots, \pi_{\text{int}}(\mathbf{b}_{m_2})$ are linearly independent in $\mathbb{R}^m / \mathcal{A}^\circ$ and

$$(2.1) \quad \mathcal{A} = \mathcal{A}^\circ + \mathbb{Z}\pi_{\text{int}}(\mathbf{b}_1) + \dots + \mathbb{Z}\pi_{\text{int}}(\mathbf{b}_{m_2}).$$

Given \mathcal{L} and a bounded subset $\mathcal{W} \subset \mathcal{A}$ with non-empty interior, we define

$$(2.2) \quad \mathcal{P}(\mathcal{W}, \mathcal{L}) = \{\pi(\mathbf{y}) : \mathbf{y} \in \mathcal{L}, \pi_{\text{int}}(\mathbf{y}) \in \mathcal{W}\} \subset \mathbb{R}^d.$$

We will call $\mathcal{P} = \mathcal{P}(\mathcal{W}, \mathcal{L})$ a *cut-and-project set*, and \mathcal{W} the *window*. We denote by $\mu_{\mathcal{A}}$ the Haar measure of \mathcal{A} , normalized so that its restriction to \mathcal{A}° is the standard m_1 -dimensional Lebesgue measure. If \mathcal{W} has boundary of measure zero with respect to $\mu_{\mathcal{A}}$, we will say $\mathcal{P}(\mathcal{W}, \mathcal{L})$ is *regular*. Set $\mathcal{V} = \mathbb{R}^d \times \mathcal{A}^\circ$; then $\mathcal{L}_{\mathcal{V}} = \mathcal{L} \cap \mathcal{V}$ is a lattice of full rank in \mathcal{V} . Let $\mu_{\mathcal{V}} = \text{vol} \times \mu_{\mathcal{A}}$ be the natural volume measure on $\mathbb{R}^d \times \mathcal{A}$ (this restricts to the standard $d + m_1$ dimensional Lebesgue measure on \mathcal{V}). It follows from Weyl equidistribution (see [6] or [10, Prop. 3.2]) that for any regular cut-and-project set \mathcal{P} and any bounded $\mathcal{D} \subset \mathbb{R}^d$ with boundary of measure zero with respect to Lebesgue measure,

$$(2.3) \quad \lim_{T \rightarrow \infty} \frac{\#\{\mathbf{b} \in \mathcal{L} : \pi(\mathbf{b}) \in \mathcal{P} \cap T\mathcal{D}\}}{T^d} = C_{\mathcal{P}} \text{vol}(\mathcal{D})$$

where

$$(2.4) \quad C_{\mathcal{P}} := \frac{\mu_{\mathcal{A}}(\mathcal{W})}{\mu_{\mathcal{V}}(\mathcal{V}/\mathcal{L}_{\mathcal{V}})}.$$

A further condition often imposed in the quasicrystal literature is that $\pi|_{\mathcal{L}}$ is injective (i.e., the map $\mathcal{L} \rightarrow \pi(\mathcal{L})$ is one-to-one); we will not require this here. To avoid coincidences in \mathcal{P} , we assume throughout this paper that the window is appropriately chosen so that the map $\pi_{\mathcal{W}} : \{\mathbf{y} \in \mathcal{L} : \pi_{\text{int}}(\mathbf{y}) \in \mathcal{W}\} \rightarrow \mathcal{P}$ is bijective. Then (2.3) implies

$$(2.5) \quad \lim_{T \rightarrow \infty} \frac{\#(\mathcal{P} \cap T\mathcal{D})}{T^d} = C_{\mathcal{P}} \text{vol}(\mathcal{D}),$$

i.e., \mathcal{P} has density $\theta(\mathcal{P}) = C_{\mathcal{P}}$. Under the above assumptions $\mathcal{P}(\mathcal{W}, \mathcal{L})$ is a Delone set, i.e., uniformly discrete and relatively dense in \mathbb{R}^d .

We furthermore extend the definition of cut-and-project sets $\mathcal{P}(\mathcal{W}, \mathcal{L})$ to affine lattices $\mathcal{L} = \mathcal{L}_0 + \mathbf{x}$ with $\mathbf{x} \in \mathbb{R}^n$ and \mathcal{L}_0 a lattice; note that $\mathcal{P}(\mathcal{W}, \mathcal{L} + \mathbf{x}) = \mathcal{P}(\mathcal{W} - \pi_{\text{int}}(\mathbf{x}), \mathcal{L}) + \pi(\mathbf{x})$.

3. RANDOM CUT-AND-PROJECT SETS

Following our approach in [10], we will now, for any given regular cut-and-project set $\mathcal{P} = \mathcal{P}(\mathcal{W}, \mathcal{L})$, construct two $\text{SL}(d, \mathbb{R})$ -invariant random point processes on \mathbb{R}^d which will describe the limit distributions in Theorem 2. Let $G = \text{ASL}(n, \mathbb{R}) = \text{SL}(n, \mathbb{R}) \ltimes \mathbb{R}^n$, with multiplication law

$$(3.1) \quad (M, \boldsymbol{\xi})(M', \boldsymbol{\xi}') = (MM', \boldsymbol{\xi}M' + \boldsymbol{\xi}').$$

Also set $\Gamma = \text{ASL}(n, \mathbb{Z}) \subset G$. Choose $g \in G$ and $\delta > 0$ so that $\mathcal{L} = \delta^{1/n}(\mathbb{Z}^n g)$, and let φ_g be the embedding of $\text{ASL}(d, \mathbb{R})$ in G given by

$$(3.2) \quad \varphi_g : \text{ASL}(d, \mathbb{R}) \rightarrow G, \quad (A, \mathbf{x}) \mapsto g \left(\begin{pmatrix} A & 0 \\ 0 & 1_m \end{pmatrix}, (\mathbf{x}, \mathbf{0}) \right) g^{-1}.$$

It then follows from Ratner's work [17, 18] that there exists a unique closed connected subgroup H_g of G such that $\Gamma \cap H_g$ is a lattice in H_g , $\varphi_g(\text{SL}(d, \mathbb{R})) \subset H_g$, and the closure of $\Gamma \backslash \Gamma \varphi_g(\text{SL}(d, \mathbb{R}))$ in $\Gamma \backslash G$ is given by

$$(3.3) \quad X = \Gamma \backslash \Gamma H_g.$$

Note that X can be naturally identified with the homogeneous space $(\Gamma \cap H_g) \backslash H_g$. We denote the unique right- H_g invariant probability measure on either of these spaces by μ ; sometimes we will also let μ denote the corresponding Haar measure on H_g . For each $x = \Gamma h \in X$ we set

$$(3.4) \quad \mathcal{P}^x := \mathcal{P}(\mathcal{W}, \delta^{1/n}(\mathbb{Z}^n h g))$$

and denote by $\widehat{\mathcal{P}}^x$ the corresponding set of visible points. Both sets are well defined since $\pi_{\text{int}}(\delta^{1/n}(\mathbb{Z}^n h g)) \subset \mathcal{A}$ for all $h \in H_g$; in fact $\pi_{\text{int}}(\delta^{1/n}(\mathbb{Z}^n h g)) = \mathcal{A}$ for μ -almost all $h \in H_g$; cf. [10, Prop. 3.5]. Note that \mathcal{P}^x and $\widehat{\mathcal{P}}^x$ with x random in (X, μ) define random point processes on \mathbb{R}^d . The fact that $\varphi_g(\text{SL}(d, \mathbb{R})) \subset H_g$ implies that these processes are $\text{SL}(d, \mathbb{R})$ -invariant.

Theorem 4. *The limit distributions in Theorem 2 are given by*

$$(3.5) \quad E(r, \sigma, \mathcal{P}) = \mu(\{x \in X : \#(\mathcal{P}^x \cap \mathfrak{C}(\sigma)) = r\})$$

and

$$(3.6) \quad E(r, \sigma, \widehat{\mathcal{P}}) = \mu(\{x \in X : \#(\widehat{\mathcal{P}}^x \cap \mathfrak{C}(\kappa_{\mathcal{P}}^{-1} \sigma)) = r\})$$

where

$$(3.7) \quad \mathfrak{C}(\sigma) = \left\{ (x_1, \dots, x_d) \in \mathbb{R}^d : 0 < x_1 < 1, \|(x_2, \dots, x_d)\| < \left(\frac{\sigma d}{C_{\mathcal{P}} v_{d-1}} \right)^{1/(d-1)} x_1 \right\}.$$

We note that relation (3.5) is a special case of [10, Thm. A.1]. The new result of the present study is (3.6).

In [10, Section 1.4] we also consider the closed connected subgroup \widetilde{H}_g of G such that $\Gamma \cap \widetilde{H}_g$ is a lattice in \widetilde{H}_g , $\varphi_g(\text{ASL}(d, \mathbb{R})) \subset \widetilde{H}_g$, and the closure of $\Gamma \backslash \Gamma \varphi_g(\text{ASL}(d, \mathbb{R}))$ in $\Gamma \backslash G$ is given by $\widetilde{X} := \Gamma \backslash \Gamma \widetilde{H}_g$. The unique right- \widetilde{H}_g invariant probability measure on \widetilde{X} is denoted by $\widetilde{\mu}$. The point process \mathcal{P}^x in (3.4) with x random in $(\widetilde{X}, \widetilde{\mu})$ is now $\text{ASL}(d, \mathbb{R})$ -invariant, i.e., in addition to the previous $\text{SL}(d, \mathbb{R})$ -invariance we also have translation-invariance. The latter implies that $\mathcal{P}^x = \widehat{\mathcal{P}}^x$ for $\widetilde{\mu}$ -almost every $x \in \widetilde{X}$. Proposition 4.5 in [10] shows that for Lebesgue-almost all $\mathbf{y} \in \mathbb{R}^d \times \{\mathbf{0}\}$ we have $H_{g(1_n, \mathbf{y})} = \widetilde{H}_g$. This has the following interesting consequence.

Corollary 5. *Given any regular cut-and-project set \mathcal{P} there is a subset $\mathfrak{S} \subset \mathbb{R}^d$ of Lebesgue measure zero such that for every $\mathbf{y} \in \mathbb{R}^d \setminus \mathfrak{S}$*

$$(3.8) \quad E(r, \sigma, \mathcal{P} + \mathbf{y}) = E(r, \sigma, \widehat{\mathcal{P}} + \mathbf{y}) = \widetilde{\mu}(\{x \in \widetilde{X} : \#(\mathcal{P}^x \cap \mathfrak{C}(\sigma)) = r\}).$$

That is, all limit distributions are independent of \mathbf{y} for Lebesgue-almost every \mathbf{y} .

4. THE SIEGEL-VEECH FORMULA FOR VISIBLE POINTS

Throughout the remaining sections, we let $\mathcal{P} = \mathcal{P}(\mathcal{W}, \mathcal{L})$ be a given regular cut-and-project set. We fix $g \in G$ and $\delta > 0$ so that $\mathcal{L} = \delta^{1/n}(\mathbb{Z}^n g)$. In fact, by an appropriate scaling of the length units, we can assume without loss of generality that $\delta = 1$. This assumption will be in force throughout the remaining sections except the last one. Hence we now have $\mathcal{P} = \mathcal{P}(\mathcal{W}, \mathbb{Z}^n g)$ and $\mathcal{P}^x = \mathcal{P}(\mathcal{W}, \mathbb{Z}^n h g)$ for each $x = \Gamma h \in X$.

The following Siegel-Veech formulas will serve as a crucial technical tool in our proofs of the main theorems.

Theorem 6. *For any $f \in L^1(\mathbb{R}^d)$,*

$$(4.1) \quad \int_X \sum_{\mathbf{q} \in \mathcal{P}^x} f(\mathbf{q}) d\mu(x) = C_{\mathcal{P}} \int_{\mathbb{R}^d} f(\mathbf{x}) d\mathbf{x}$$

and

$$(4.2) \quad \int_X \sum_{\mathbf{q} \in \widehat{\mathcal{P}}^x} f(\mathbf{q}) d\mu(x) = \kappa_{\mathcal{P}} C_{\mathcal{P}} \int_{\mathbb{R}^d} f(\mathbf{x}) d\mathbf{x}.$$

Veech has proved formulas of the above type for general $\mathrm{SL}(d, \mathbb{R})$ -invariant measures [22, Thm. 0.12]. The proof of Theorem 6 is simpler in the present setting. Relation (4.1) was proved in [10, Theorem 1.5]. In the present section we will prove that *there exists* $0 < \kappa_{\mathcal{P}} \leq 1$ such that relation (4.2) holds for all $f \in L^1(\mathbb{R}^d)$. We will then later establish that this $\kappa_{\mathcal{P}}$ indeed yields the relative density defined in (1.6).

Consider the map

$$(4.3) \quad B \mapsto \int_X \#(\widehat{\mathcal{P}}^x \cap B) d\mu(x) \quad (B \text{ any Borel subset of } \mathbb{R}^d).$$

This map defines a Borel measure on \mathbb{R}^d , which is finite on any compact set B (by [10, Theorem 1.5]), invariant under $\mathrm{SL}(d, \mathbb{R})$, and gives zero point mass to $\mathbf{0} \in \mathbb{R}^d$. Hence up to a constant, the measure must equal Lebesgue measure, i.e. there exists a constant $\kappa_{\mathcal{P}} \geq 0$ such that

$$(4.4) \quad \int_X \#(\widehat{\mathcal{P}}^x \cap B) d\mu(x) = \kappa_{\mathcal{P}} C_{\mathcal{P}} \mathrm{vol}(B)$$

for every Borel set $B \subset \mathbb{R}^d$. By a standard approximation argument, this implies that (4.2) holds for all $f \in L^1(\mathbb{R}^d)$. Also $\kappa_{\mathcal{P}} \leq 1$ is immediate from (4.1).

It remains to verify that $\kappa_{\mathcal{P}} > 0$. Recall that we are assuming that \mathcal{W} has non-empty interior \mathcal{W}° in $\mathcal{A} = \pi_{\mathrm{int}}(\mathcal{L})$. Now take B to be any bounded open set in \mathbb{R}^d which is star-shaped with center $\mathbf{0}$ and such that $(B \setminus \{\mathbf{0}\}) \times \mathcal{W}^\circ$ contains some point in the (affine) lattice \mathcal{L} . Then the set of $x = \Gamma h$ in X for which $\mathbb{Z}^n h g$ has at least one point in $(B \setminus \{\mathbf{0}\}) \times \mathcal{W}^\circ$ is non-empty and open. Note that for any such x , $\mathcal{P}^x = \mathcal{P}(\mathcal{W}, \mathbb{Z}^n h g)$ has a point in $B \setminus \{\mathbf{0}\}$, and hence also a *visible* point in $B \setminus \{\mathbf{0}\}$, since B is star-shaped. It follows that the left hand side of (4.4) is positive for our set B . Therefore $\kappa_{\mathcal{P}} > 0$, as claimed.

5. THE LIMIT DISTRIBUTION FOR SMALL σ

From now on we take $E(r, \sigma, \mathcal{P})$ and $E(r, \sigma, \widehat{\mathcal{P}})$ to be defined by the relations (3.5), (3.6). Then (1.7) holds by [10, Thm. A.1], and we will prove in Section 10 that also (1.8) holds.

In the present section we will prove that the relation (1.9),

$$(5.1) \quad E(0, \sigma, \mathcal{P}) = 1 - \kappa_{\mathcal{P}} \sigma + o(\sigma),$$

holds with the same $\kappa_{\mathcal{P}} \in (0, 1]$ as in the Siegel-Veech formula (4.2). Rel. (1.10) is then a simple consequence of the observation that

$$(5.2) \quad E(0, \sigma, \widehat{\mathcal{P}}) = E(0, \kappa_{\mathcal{P}}^{-1} \sigma, \mathcal{P}).$$

To prove (5.1), first note that, for any $\sigma > 0$,

$$(5.3) \quad \begin{aligned} 1 - E(0, \sigma, \mathcal{P}) &= \mu(\{x \in X : \mathcal{P}^x \cap \mathfrak{C}(\sigma) \neq \emptyset\}) = \mu(\{x \in X : \widehat{\mathcal{P}}^x \cap \mathfrak{C}(\sigma) \neq \emptyset\}) \\ &\leq \int_X \#(\widehat{\mathcal{P}}^x \cap \mathfrak{C}(\sigma)) d\mu(x) = \kappa_{\mathcal{P}} C_{\mathcal{P}} \mathrm{vol}(\mathfrak{C}(\sigma)) = \kappa_{\mathcal{P}} \sigma, \end{aligned}$$

where the integral was evaluated using (4.4).

On the other hand using the fact that the point process \mathcal{P}^x ($x \in (X, \mu)$) is invariant under $\mathrm{SO}(d)$, and $\widehat{\mathcal{P}}'k = \widehat{\mathcal{P}}'k$ for every point set \mathcal{P}' and every $k \in \mathrm{SO}(d)$, we have

$$(5.4) \quad 1 - E(0, \sigma, \mathcal{P}) = \int_X A(\sigma, \mathcal{P}^x) d\mu(x)$$

with

$$(5.5) \quad A(\sigma, \mathcal{P}^x) = \int_{\mathrm{SO}(d)} I(\widehat{\mathcal{P}}^x \cap \mathfrak{C}(\sigma)k \neq \emptyset) dk,$$

where dk is Haar measure on $\mathrm{SO}(d)$ normalized by $\int_{\mathrm{SO}(d)} dk = 1$.

We write $\varphi(\mathbf{p}, \mathbf{q}) \in [0, \pi]$ for the angle between any two points $\mathbf{p}, \mathbf{q} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$, as seen from $\mathbf{0}$. Also for any $x \in X$ we set

$$(5.6) \quad \sigma_0(\mathcal{P}^x) = \frac{C_{\mathcal{P}} v_{d-1}}{d} \left(\tan \frac{\varphi_0(\mathcal{P}^x)}{2} \right)^{d-1}$$

where

$$(5.7) \quad \varphi_0(\mathcal{P}^x) = \min\{\varphi(\mathbf{p}, \mathbf{q}) : \mathbf{p}, \mathbf{q} \in \widehat{\mathcal{P}}^x \cap \mathcal{B}_1^d, \mathbf{p} \neq \mathbf{q}\},$$

with the convention that $\varphi_0(\mathcal{P}^x) = \pi$ and $\sigma_0(\mathcal{P}^x) = +\infty$ whenever $\#(\widehat{\mathcal{P}}^x \cap \mathcal{B}_1^d) \leq 1$. These are measurable functions on X , and $\varphi_0(\mathcal{P}^x) > 0$ and $\sigma_0(\mathcal{P}^x) > 0$ for all $x \in X$.

Now if $0 < \sigma < \sigma_0(\mathcal{P}^x)$ then for any two distinct points $\mathbf{p}, \mathbf{q} \in \widehat{\mathcal{P}}^x \cap \mathcal{B}_1^d$ we have

$$(5.8) \quad \varphi(\mathbf{p}, \mathbf{q}) > 2 \arctan \left(\left(\frac{\sigma d}{C_{\mathcal{P}} v_{d-1}} \right)^{1/(d-1)} \right),$$

and because of the definition of $\mathfrak{C}(\sigma)$, (3.7), this implies that there does not exist any $k \in \text{SO}(d)$ for which $\mathfrak{C}(\sigma)k$ contains both \mathbf{p} and \mathbf{q} . Hence for $0 < \sigma < \sigma_0(\mathcal{P}^x)$ we have (writing $\mathbf{e}_1 = (1, 0, \dots, 0) \in \mathbb{R}^d$)

$$(5.9) \quad \begin{aligned} A(\sigma, \mathcal{P}^x) &\geq \sum_{\mathbf{p} \in \widehat{\mathcal{P}}^x \cap \mathcal{B}_1^d} \int_{\text{SO}(d)} I(\mathbf{p} \in \mathfrak{C}(\sigma)k) dk = \#(\widehat{\mathcal{P}}^x \cap \mathcal{B}_1^d) \cdot \int_{\text{SO}(d)} I(\mathbf{e}_1 \in \mathfrak{C}(\sigma)k) dk \\ &= \frac{\text{vol}(\mathfrak{C}(\sigma) \cap \mathcal{B}_1^d)}{\text{vol}(\mathcal{B}_1^d)} \#(\widehat{\mathcal{P}}^x \cap \mathcal{B}_1^d), \end{aligned}$$

and here

$$(5.10) \quad \frac{\text{vol}(\mathfrak{C}(\sigma) \cap \mathcal{B}_1^d)}{\text{vol}(\mathcal{B}_1^d)} \sim \frac{\text{vol}(\mathfrak{C}(\sigma))}{\text{vol}(\mathcal{B}_1^d)} = \frac{\sigma}{v_d C_{\mathcal{P}}} \quad \text{as } \sigma \rightarrow 0.$$

Hence given any number $K < (v_d C_{\mathcal{P}})^{-1}$, there is some $\sigma(K) > 0$ such that for all $0 < \sigma < \sigma(K)$ we have

$$(5.11) \quad 1 - E(0, \sigma, \mathcal{P}) = \int_X A(\sigma, \mathcal{P}^x) d\mu(x) \geq K\sigma \int_X I(\sigma < \sigma_0(\mathcal{P}^x)) \#(\widehat{\mathcal{P}}^x \cap \mathcal{B}_1^d) d\mu(x).$$

Furthermore, by the Monotone Convergence Theorem and (4.4),

$$(5.12) \quad \lim_{\sigma \rightarrow 0} \int_X I(\sigma < \sigma_0(\mathcal{P}^x)) \#(\widehat{\mathcal{P}}^x \cap \mathcal{B}_1^d) d\mu(x) = \int_X \#(\widehat{\mathcal{P}}^x \cap \mathcal{B}_1^d) d\mu(\mathcal{P}^x) = \kappa_{\mathcal{P}} C_{\mathcal{P}} v_d.$$

We thus conclude

$$(5.13) \quad \liminf_{\sigma \rightarrow 0} \frac{1 - E(0, \sigma, \mathcal{P})}{\sigma} \geq K \kappa_{\mathcal{P}} C_{\mathcal{P}} v_d.$$

The claim (5.1) follows from (5.3) and the fact that (5.13) holds for every $K < (v_d C_{\mathcal{P}})^{-1}$.

6. LOWER BOUND ON THE DENSITY OF VISIBLE POINTS

Combining (5.1) and (1.7) (recall that the latter was proved in [10, Thm. A.1]), we get the following lower bound on the density $\theta(\widehat{\mathcal{P}}) = \kappa_{\mathcal{P}} C_{\mathcal{P}}$ in Theorem 1:

Lemma 7. *Let \mathfrak{U} be any subset of S_1^{d-1} with boundary of measure zero (w.r.t. ω), and let $\mathcal{D} = \{\mathbf{v} \in \mathbb{R}^d : 0 < \|\mathbf{v}\| < 1, \|\mathbf{v}\|^{-1}\mathbf{v} \in \mathfrak{U}\}$ be the corresponding sector in \mathcal{B}_1^d . Then*

$$(6.1) \quad \liminf_{T \rightarrow \infty} \frac{\#(\widehat{\mathcal{P}} \cap T\mathcal{D})}{T^d} \geq \kappa_{\mathcal{P}} C_{\mathcal{P}} \text{vol}(\mathcal{D}).$$

Proof. We may assume $\omega(\mathfrak{U}) > 0$, since otherwise $\text{vol}(\mathcal{D}) = 0$ and the lemma is trivial. Let $\varepsilon > 0$ be given, and let $\mathfrak{U}_\varepsilon^- \subset S_1^{d-1}$ be the “ ε -thinning” of \mathfrak{U} , that is

$$(6.2) \quad \mathfrak{U}_\varepsilon^- = \{\mathbf{v} \in S_1^{d-1} : [\varphi(\mathbf{w}, \mathbf{v}) < \varepsilon \Rightarrow \mathbf{w} \in \mathfrak{U}], \forall \mathbf{w} \in S_1^{d-1}\}.$$

(Recall that $\varphi(\mathbf{w}, \mathbf{v}) \in [0, \pi]$ is the angle between \mathbf{w} and \mathbf{v} as seen from $\mathbf{0}$.) Then $\omega(\mathfrak{U}_\varepsilon^-) \rightarrow \omega(\mathfrak{U})$ as $\varepsilon \rightarrow 0$, since \mathfrak{U} by assumption is a Jordan measurable subset of S_1^{d-1} . From now on we assume that ε is so small that $\omega(\mathfrak{U}_\varepsilon^-) > 0$. We let λ be ω restricted to $\mathfrak{U}_\varepsilon^-$ and normalized to be a probability measure; thus $\lambda(B) = \omega(\mathfrak{U}_\varepsilon^-)^{-1} \omega(B \cap \mathfrak{U}_\varepsilon^-)$ for any Borel subset $B \subset S_1^{d-1}$.

Now note that, by the definitions of $\mathcal{N}_T(\sigma, \mathbf{v}, \mathcal{P})$ and $\widehat{\mathcal{P}}$, for any $\sigma > 0$, $T > 0$ and $\mathbf{v} \in S_1^{d-1}$ we have $\mathcal{N}_T(\sigma, \mathbf{v}, \mathcal{P}) > 0$ if and only if there is some $\mathbf{y} \in \widehat{\mathcal{P}} \cap \mathcal{B}_T^d$ such that $\|\mathbf{y}\|^{-1} \mathbf{y} \in \mathcal{D}_T(\sigma, \mathbf{v})$. Furthermore, if T is larger than a certain constant depending on $\sigma, \mathcal{P}, \varepsilon$, then $\mathcal{D}_T(\sigma, \mathbf{v}) \subset \mathfrak{U}$ for every $\mathbf{v} \in \mathfrak{U}_\varepsilon^-$, meaning that $\|\mathbf{y}\|^{-1} \mathbf{y} \in \mathcal{D}_T(\sigma, \mathbf{v})$ implies $\mathbf{y} \in \mathbb{R}_{>0} \mathcal{D}$. Hence for such T and σ we have

$$(6.3) \quad \begin{aligned} \lambda(\{\mathbf{v} \in S_1^{d-1} : \mathcal{N}_T(\sigma, \mathbf{v}, \mathcal{P}) > 0\}) &= \lambda(\{\mathbf{v} \in S_1^{d-1} : [\exists \mathbf{y} \in \widehat{\mathcal{P}} \cap \mathcal{B}_T^d : \|\mathbf{y}\|^{-1} \mathbf{y} \in \mathcal{D}_T(\sigma, \mathbf{v})]\}) \\ &\leq \sum_{\mathbf{y} \in \widehat{\mathcal{P}} \cap T\mathcal{D}} \lambda(\{\mathbf{v} \in S_1^{d-1} : \|\mathbf{y}\|^{-1} \mathbf{y} \in \mathcal{D}_T(\sigma, \mathbf{v})\}) \leq \frac{\omega(\mathcal{D}_T(\sigma, \mathbf{e}_1))}{\omega(\mathfrak{U}_\varepsilon^-)} \cdot \#(\widehat{\mathcal{P}} \cap T\mathcal{D}) \\ &= \frac{\sigma d}{\omega(\mathfrak{U}_\varepsilon^-) C_{\mathcal{P}} T^d} \cdot \#(\widehat{\mathcal{P}} \cap T\mathcal{D}). \end{aligned}$$

Hence, letting $T \rightarrow \infty$ and applying (1.7) we have, for any fixed $\sigma > 0$,

$$(6.4) \quad \liminf_{T \rightarrow \infty} \frac{\#\widehat{\mathcal{P}} \cap T\mathcal{D}}{T^d} \geq \frac{\omega(\mathfrak{U}_\varepsilon^-) C_{\mathcal{P}}}{d} \cdot \frac{1 - E(0, \sigma, \mathcal{P})}{\sigma}.$$

Letting $\sigma \rightarrow 0$ in the right hand side and using (5.1), this gives

$$(6.5) \quad \liminf_{T \rightarrow \infty} \frac{\#\widehat{\mathcal{P}} \cap T\mathcal{D}}{T^d} \geq \kappa_{\mathcal{P}} C_{\mathcal{P}} \frac{\omega(\mathfrak{U}_\varepsilon^-)}{d}.$$

Finally letting $\varepsilon \rightarrow 0$ and using $\omega(\mathfrak{U})/d = \text{vol}(\mathcal{D})$ we obtain the statement of the lemma. \square

7. CONTINUITY IN THE SPACE OF CUT-AND-PROJECT SETS

Next, in Lemma 9 and Lemma 10, we will prove that for almost all $x \in X$, both \mathcal{P}^x and $\widehat{\mathcal{P}}^x$ vary continuously as we perturb x .

Lemma 8. *For any $\mathbf{m} \in \mathbb{R}^n$, if $\pi(\mathbf{m}hg) \neq \mathbf{0}$ for some $h \in H_g$ then $\pi(\mathbf{m}hg) \neq \mathbf{0}$ for μ -almost all $h \in H_g$. Similarly, for any $\mathbf{m}, \mathbf{n} \in \mathbb{R}^n$, if $\dim \text{Span}\{\pi(\mathbf{n}hg), \pi(\mathbf{m}hg)\} = 2$ for some $h \in H_g$ then $\dim \text{Span}\{\pi(\mathbf{n}hg), \pi(\mathbf{m}hg)\} = 2$ for μ -almost all $h \in H_g$.*

Proof. H_g is a connected, real-analytic manifold; hence any real-analytic function on H_g which does not vanish identically is non-zero almost everywhere. The first part of the lemma follows by applying this principle to the coordinate functions $h \mapsto \pi(\mathbf{m}hg) \cdot \mathbf{e}_j$ for $j = 1, \dots, d$. The second part of the lemma follows by applying the same principle to the functions

$$(7.1) \quad h \mapsto (\pi(\mathbf{m}hg) \cdot \mathbf{e}_i)(\pi(\mathbf{n}hg) \cdot \mathbf{e}_j) - (\pi(\mathbf{m}hg) \cdot \mathbf{e}_j)(\pi(\mathbf{n}hg) \cdot \mathbf{e}_i),$$

for $1 \leq i < j \leq d$. \square

Lemma 9. *For μ -almost every $x \in X$, and for every bounded open set $U \subset \mathbb{R}^d$ with $\mathcal{P}^x \cap \partial U = \emptyset$, there is an open set $\Omega \subset X$ with $x \in \Omega$ such that $\#(\mathcal{P}^{x'} \cap U) = \#(\mathcal{P}^x \cap U)$ for all $x' \in \Omega$.*

Proof. For each $\mathbf{m} \in \mathbb{Z}^n$, by an argument as in Lemma 8 we either have $\mathbf{m}hg \neq \mathbf{0}$ for almost all $h \in H_g$ or else $\mathbf{m}hg = \mathbf{0}$ for all $h \in H_g$. By taking $h = 1$ we see that the latter property can hold for at most one $\mathbf{m} \in \mathbb{Z}^n$, and if it holds then we necessarily have $\mathbf{m} = \mathbf{0}g^{-1}$, and $H_g \subset g\text{SL}(n, \mathbb{R})g^{-1}$. If such an exceptional \mathbf{m} exists we call it \mathbf{m}_E , and we set $(\mathbb{Z}^n)' := \mathbb{Z}^n \setminus \{\mathbf{m}_E\}$; otherwise we set $(\mathbb{Z}^n)' := \mathbb{Z}^n$.

Now consider the following two subsets of H_g :

$$(7.2) \quad S_1 = \{h \in H_g : (\mathbb{Z}^n)'hg \cap (\mathbb{R}^d \times \partial\mathcal{W}) \neq \emptyset\};$$

$$(7.3) \quad S_2 = \{h \in H_g : \exists \ell_1 \neq \ell_2 \in \mathbb{Z}^n hg \cap \pi_{\text{int}}^{-1}(\mathcal{W}) \text{ satisfying } \pi(\ell_1) = \pi(\ell_2)\}.$$

We have $\mu(S_1) = 0$, by [10, Theorem 5.1]. Also $\mu(S_2) = 0$, by [10, Prop. 3.7] applied to \mathcal{W}° . We will prove the lemma by showing that for every $h \in H_g \setminus (S_1 \cup S_2)$, the point $x = \Gamma h \in X$ has the property described in the lemma.

Thus let $h \in H_g \setminus (S_1 \cup S_2)$ be given, set $x = \Gamma h \in X$, and let U be an arbitrary bounded open subset of \mathbb{R}^d with boundary disjoint from $\mathcal{P}^x = \mathcal{P}(\mathcal{W}, \mathbb{Z}^n hg)$. Assume that the desired property does *not* hold. Then there is a sequence h_1, h_2, \dots in H_g tending to h such that

$$(7.4) \quad \#(\mathcal{P}(\mathcal{W}, \mathbb{Z}^n h_j g) \cap U) \neq \#(\mathcal{P}(\mathcal{W}, \mathbb{Z}^n hg) \cap U), \quad \forall j.$$

Let F be the (finite) set

$$(7.5) \quad F = \{\mathbf{m} \in \mathbb{Z}^n : \mathbf{m}hg \in U \times \mathcal{W}\}.$$

Note that $\mathbf{m}hg \in U \times \mathcal{W}^\circ$ for every $\mathbf{m} \in F \cap (\mathbb{Z}^n)'$, since $h \notin S_1$. But $U \times \mathcal{W}^\circ$ is open; hence by continuity we also have $\mathbf{m}h'g \in U \times \mathcal{W}^\circ$ for every $h' \in H_g$ sufficiently near h and all $\mathbf{m} \in F \cap (\mathbb{Z}^n)'$. Note also that if the exceptional point \mathbf{m}_E exists and belongs to F then $\mathbf{0} = \mathbf{m}_E h'g \in U \times \mathcal{W}$ for all $h' \in H_g$. Hence, for every $h' \in H_g$ near h we have

$$(7.6) \quad \mathcal{P}(\mathcal{W}, \mathbb{Z}^n h'g) \supset \{\pi(\mathbf{m}h'g) : \mathbf{m} \in F\}.$$

Because of $h \notin S_2$, the points $\pi(\mathbf{m}hg)$ for $\mathbf{m} \in F$ are pairwise distinct. By continuity it then also follows that for any $h' \in H_g$ sufficiently near h , the points $\pi(\mathbf{m}h'g)$ for $\mathbf{m} \in F$ are pairwise distinct. Hence $\#(\mathcal{P}(\mathcal{W}, \mathbb{Z}^n hg) \cap U) = \#F$ and $\#(\mathcal{P}(\mathcal{W}, \mathbb{Z}^n h'g) \cap U) \geq \#F$ for every h' near h . Therefore in (7.4), the left hand side must be *larger* than $\#F$, for all large j . Hence for each large j there is some $\mathbf{m} \in \mathbb{Z}^n \setminus F$ such that $\mathbf{m}h_j g \in U \times \mathcal{W}$. But for any compact $C \subset H_g$ the set $\cup_{h' \in C} (U \times \mathcal{W})g^{-1}h'^{-1}$ is bounded and hence has finite intersection with \mathbb{Z}^n . Therefore there is a bounded number of possibilities for \mathbf{m} as j varies, and by passing to a subsequence we may assume that \mathbf{m} is independent of j .

Now for our fixed $\mathbf{m} \in \mathbb{Z}^n \setminus F$ we have $\mathbf{m}h_j g \in U \times \mathcal{W}$ for all j , but $\mathbf{m}h_j g \rightarrow \mathbf{m}hg \notin U \times \mathcal{W}$ as $j \rightarrow \infty$; this forces $\mathbf{m}hg \in \partial(U \times \mathcal{W})$, and it also implies that we cannot have $\mathbf{m} = \mathbf{m}_E$. But $\pi_{\text{int}}(\mathbf{m}hg) \notin \partial\mathcal{W}$ since $h \notin S_1$, and thus we must have $\pi(\mathbf{m}hg) \in \partial U$. Note also that $\pi_{\text{int}}(\mathbf{m}hg)$ cannot belong to the exterior of \mathcal{W} , since then the same would hold for $\pi_{\text{int}}(\mathbf{m}h_j g)$ for j large, contradicting $\mathbf{m}h_j g \in U \times \mathcal{W}$. Hence $\pi_{\text{int}}(\mathbf{m}hg)$ must belong to the interior of \mathcal{W} ; therefore $\pi(\mathbf{m}hg) \in \mathcal{P}^x = \mathcal{P}(\mathcal{W}, \mathbb{Z}^n hg)$. This contradicts our assumption that \mathcal{P}^x is disjoint from ∂U , and so the lemma is proved. \square

Lemma 10. *For μ -almost every $x \in X$, and for every bounded open set $U \subset \mathbb{R}^d$ with $\widehat{\mathcal{P}}^x \cap \partial U = \emptyset$, there is an open set $\Omega \subset X$ with $x \in \Omega$ such that $\#(\widehat{\mathcal{P}}^{x'} \cap U) = \#(\widehat{\mathcal{P}}^x \cap U)$ for all $x' \in \Omega$.*

Proof. Let $\mathbf{m}_E, (\mathbb{Z}^n)', S_1$ and S_2 be as in the proof of Lemma 9. Also set

$$S_3 = \{h \in H_g : \exists \mathbf{m} \in \mathbb{Z}^n, h' \in H_g \text{ satisfying } \pi(\mathbf{m}hg) = \mathbf{0}, \pi(\mathbf{m}h'g) \neq \mathbf{0}\}$$

$$S_4 = \{h \in H_g : \exists \mathbf{m}, \mathbf{n} \in \mathbb{Z}^n, h' \in H_g \text{ satisfying } \dim \text{Span}\{\pi(\mathbf{n}hg), \pi(\mathbf{m}hg)\} \leq 1 \\ \text{and } \dim \text{Span}\{\pi(\mathbf{n}h'g), \pi(\mathbf{m}h'g)\} = 2\}.$$

Using Lemma 8 and the fact that \mathbb{Z}^n is countable, we have $\mu(S_3) = \mu(S_4) = 0$.

Now let $h \in H_g \setminus (S_1 \cup S_2 \cup S_3 \cup S_4)$ be given, set $x = \Gamma h \in X$, and let U be an arbitrary bounded open subset of \mathbb{R}^d with boundary disjoint from $\widehat{\mathcal{P}}^x = \widehat{\mathcal{P}}(\mathcal{W}, \mathbb{Z}^n hg)$. Assume that there is a sequence h_1, h_2, \dots in H_g tending to h such that

$$(7.7) \quad \#(\widehat{\mathcal{P}}(\mathcal{W}, \mathbb{Z}^n h_j g) \cap U) \neq \#(\widehat{\mathcal{P}}(\mathcal{W}, \mathbb{Z}^n hg) \cap U), \quad \forall j.$$

We will show that this leads to a contradiction, and this will complete the proof of the lemma (cf. the proof of Lemma 9).

As an initial reduction, let us note that we may assume $\mathcal{P}^x \cap \partial U = \emptyset$. Indeed, recall that \mathcal{P}^x is locally finite (cf. [10, Prop. 3.1]); hence the set $A = \mathcal{P}^x \cap \partial U$ is certainly finite. Also every point in A is invisible in \mathcal{P}^x , since we are assuming $\widehat{\mathcal{P}}^x \cap \partial U = \emptyset$. If $A \neq \emptyset$ then fix $r > 0$ so small that $(\mathbf{p} + \mathcal{B}_{2r}^d) \cap \mathcal{P}^x = \{\mathbf{p}\}$ for each $\mathbf{p} \in A$, and set $U' = U \cup (\cup_{\mathbf{p} \in A} (\mathbf{p} + \mathcal{B}_r^d))$ and $U'' = U \setminus (\cup_{\mathbf{p} \in A} (\mathbf{p} + \overline{\mathcal{B}_r^d}))$. These are bounded open sets satisfying $\#(\widehat{\mathcal{P}}^x \cap U') = \#(\widehat{\mathcal{P}}^x \cap U'') = \#(\widehat{\mathcal{P}}^x \cap U)$ and $\mathcal{P}^x \cap \partial U' = \mathcal{P}^x \cap \partial U'' = \emptyset$. For each j we must have either $\#(\widehat{\mathcal{P}}(\mathcal{W}, \mathbb{Z}^n h_j g) \cap U') > \#(\widehat{\mathcal{P}}^x \cap U)$ or $\#(\widehat{\mathcal{P}}(\mathcal{W}, \mathbb{Z}^n h_j g) \cap U'') < \#(\widehat{\mathcal{P}}^x \cap U)$, because of $U'' \subset U \subset U'$ and (7.7). Hence after replacing U by U' or U'' , and passing to a subsequence, we are in a situation where (7.7) holds, and also $\mathcal{P}^x \cap \partial U = \emptyset$.

Now take F as in (7.5); it then follows from the proof of Lemma 9 that $\#(\mathcal{P}^x \cap U) = \#F$ and also $\#(\mathcal{P}(\mathcal{W}, \mathbb{Z}^n h_j g) \cap U) = \#F$ for every large j . Hence (7.7) implies that for every large j there is some $\mathbf{m} \in F$ such that either $\pi(\mathbf{m} h_j g)$ is visible in $\mathcal{P}(\mathcal{W}, \mathbb{Z}^n h_j g)$ but $\pi(\mathbf{m} h g)$ is invisible in \mathcal{P}^x , or the other way around. Since F is finite we may assume, by passing to a subsequence, that \mathbf{m} is independent of j .

First assume that $\pi(\mathbf{m} h g)$ is invisible in \mathcal{P}^x but $\pi(\mathbf{m} h_j g)$ is visible in $\mathcal{P}(\mathcal{W}, \mathbb{Z}^n h_j g)$ for every large j . In particular then $\pi(\mathbf{m} h_j g) \neq \mathbf{0}$ for large j , and since $h \notin S_3$ this implies $\pi(\mathbf{m} h g) \neq \mathbf{0}$. The invisibility of $\pi(\mathbf{m} h g)$ means that there exist $\mathbf{n} \in \mathbb{Z}^n$ and $0 < t < 1$ such that $\pi_{\text{int}}(\mathbf{n} h g) \in \mathcal{W}$ and $\pi(\mathbf{n} h g) = t\pi(\mathbf{m} h g)$. Now $\pi_{\text{int}}(\mathbf{n} h g) \in \mathcal{W}$ and $h \notin S_1$ force $\pi_{\text{int}}(\mathbf{n} h g) \in \mathcal{W}^\circ$; hence $\pi_{\text{int}}(\mathbf{n} h_j g) \in \mathcal{W}^\circ$ for all large j and so $\pi(\mathbf{n} h_j g) \in \mathcal{P}(\mathcal{W}, \mathbb{Z}^n h_j g)$. On the other hand $\dim \text{Span}\{\pi(\mathbf{n} h g), \pi(\mathbf{m} h g)\} = 1$ together with $h \notin S_4$ imply $\dim \text{Span}\{\pi(\mathbf{n} h' g), \pi(\mathbf{m} h' g)\} \leq 1$ for all $h' \in H_g$. Using also $h_j \rightarrow h$, $\pi(\mathbf{m} h g) \neq \mathbf{0}$ and $0 < t < 1$, this implies that for every large j there is $0 < t_j < 1$ such that $\pi(\mathbf{n} h_j g) = t_j \pi(\mathbf{m} h_j g)$. Hence $\pi(\mathbf{m} h_j g)$ is invisible in $\mathcal{P}(\mathcal{W}, \mathbb{Z}^n h_j g)$ for every large j , contradicting our earlier assumption.

It remains to treat the case when $\pi(\mathbf{m} h g)$ is visible in \mathcal{P}^x but $\pi(\mathbf{m} h_j g)$ is invisible in $\mathcal{P}(\mathcal{W}, \mathbb{Z}^n h_j g)$ for every large j . Then for every large j there exist $\mathbf{n} \in \mathbb{Z}^n$ and $0 < t_j < 1$ such that $\pi_{\text{int}}(\mathbf{n} h_j g) \in \mathcal{W}$ and $\pi(\mathbf{n} h_j g) = t_j \pi(\mathbf{m} h_j g)$. It is easily seen that there are only a finite number of possibilities for \mathbf{n} , and hence by passing to a subsequence we may assume that \mathbf{n} is independent of j . Since $\pi(\mathbf{m} h g)$ is visible in \mathcal{P}^x we have $\pi(\mathbf{m} h g) \neq \mathbf{0}$; hence also $\pi(\mathbf{m} h_j g) \neq \mathbf{0}$ for all large j , and this forces $\mathbf{n} \neq \mathbf{m}$. Also $\pi(\mathbf{m} h_j g) \rightarrow \pi(\mathbf{m} h g) \neq \mathbf{0}$ and $t_j \pi(\mathbf{m} h_j g) = \pi(\mathbf{n} h_j g) \rightarrow \pi(\mathbf{n} h g)$ imply that $t = \lim_{j \rightarrow \infty} t_j \in [0, 1]$ exists, and $\pi(\mathbf{n} h g) = t\pi(\mathbf{m} h g)$. Using $h \notin S_1$ and $\pi_{\text{int}}(\mathbf{n} h_j g) \in \mathcal{W}$ it follows that also $\pi_{\text{int}}(\mathbf{n} h g) \in \mathcal{W}$ and so $\pi(\mathbf{n} h g) \in \mathcal{P}^x$. Using $h \notin S_3$ and $\pi(\mathbf{n} h_j g) \neq \mathbf{0}$ for j large, it follows that $\pi(\mathbf{n} h g) \neq \mathbf{0}$; furthermore using $h \notin S_2$ we have $\pi(\mathbf{n} h g) \neq \pi(\mathbf{m} h g)$. Hence $0 < t < 1$, and so $\pi(\mathbf{m} h g)$ is invisible in \mathcal{P}^x , contradicting our earlier assumption. \square

8. UPPER BOUND ON THE DENSITY OF VISIBLE POINTS

We are now in position to prove an upper bound complementing Lemma 7.

Lemma 11. *We have $\lim_{T \rightarrow \infty} \frac{\#(\widehat{\mathcal{P}} \cap \mathcal{B}_T^d)}{T^d} = \kappa_{\mathcal{P}} C_{\mathcal{P}} v_d$.*

Proof. For any $\mathcal{P}' \subset \mathbb{R}^d$, let us write $\widetilde{\mathcal{P}}' = \mathcal{P}' \setminus \widehat{\mathcal{P}}'$ for the set of invisible points in \mathcal{P}' . Define $F : X \rightarrow \mathbb{Z}_{\geq 0}$ through

$$(8.1) \quad F(x) = \liminf_{x' \rightarrow x} \#(\widetilde{\mathcal{P}}^{x'} \cap \mathcal{B}_1^d).$$

Then F is lower semicontinuous by construction. Hence by [10, Thm. 4.1] and the Portmanteau theorem (cf., e.g., [24, Thm. 1.3.4(iv)]),

$$(8.2) \quad \liminf_{R \rightarrow \infty} \int_{\text{SO}(d)} F(\Gamma \varphi_g(k \Phi^{\log R})) dk \geq \int_X F d\mu,$$

with

$$(8.3) \quad \Phi^t = \begin{pmatrix} e^{-(d-1)t} & \mathbf{0} \\ \mathbf{0} & e^t \mathbf{1}_{d-1} \end{pmatrix} \in \mathrm{SL}(d, \mathbb{R}).$$

Now in the left hand side of (8.2), we use the fact that for any $x = \Gamma\varphi_g(T)$, $T \in \mathrm{SL}(d, \mathbb{R})$, we have

$$(8.4) \quad F(x) \leq \#(\tilde{\mathcal{P}}^x \cap \mathcal{B}_1^d) = \#(\tilde{\mathcal{P}}(\mathcal{W}, \mathbb{Z}^n \varphi_g(T)g) \cap \mathcal{B}_1^d) = \#(\tilde{\mathcal{P}} \cap \mathcal{B}_1^d T^{-1}).$$

In the right hand side of (8.2) we note that if $x = \Gamma h$ has both the continuity properties described in Lemmata 9 and 10, and if furthermore $\mathcal{P}^x \cap \mathrm{S}_1^{d-1} = \emptyset$, then in fact $F(x) = \#(\tilde{\mathcal{P}}^x \cap \mathcal{B}_1^d)$. But these conditions are fulfilled for μ -almost all $x \in X$ (concerning $\mathcal{P}^x \cap \mathrm{S}_1^{d-1} = \emptyset$, use [10, Thm. 1.5]). Hence it follows from (8.2) that

$$(8.5) \quad \liminf_{R \rightarrow \infty} \int_{\mathrm{SO}(d)} \#(\tilde{\mathcal{P}} \cap \mathcal{B}_1^d \Phi^{-\log R} k^{-1}) dk \geq \int_X \#(\tilde{\mathcal{P}}^x \cap \mathcal{B}_1^d) d\mu(x) = (1 - \kappa_{\mathcal{P}})C_{\mathcal{P}}v_d,$$

where the last equality holds by Theorem 6.

But exactly as in the proof of Theorem 5.1 in [10], we have for any $R > 1$

$$(8.6) \quad \int_{\mathrm{SO}(d)} \#(\tilde{\mathcal{P}} \cap \mathcal{B}_1 \Phi^{-\log R} k^{-1}) dk = \sum_{\mathbf{p} \in \tilde{\mathcal{P}}} A_R(\|\mathbf{p}\|) = \int_0^\infty A_R(\tau) d\tilde{N}(\tau) = - \int_0^\infty \tilde{N}(\tau) dA_R(\tau),$$

where

$$(8.7) \quad \tilde{N}(T) = \#(\tilde{\mathcal{P}} \cap \mathcal{B}_T^d),$$

and A_R is the continuous and decreasing function from $\mathbb{R}_{\geq 0}$ to $[0, 1]$ given by $A_R(0) = 1$ and

$$(8.8) \quad A_R(\tau) = \frac{\omega(\mathrm{S}_1^{d-1} \cap \tau^{-1} \mathcal{B}_1^d \Phi^{-\log R})}{\omega(\mathrm{S}_1^{d-1})} \quad \text{for } \tau > 0.$$

(Thus $A_R(\tau) = 1$ for $0 \leq \tau \leq R^{-1}$ and $A_R(\tau) = 0$ for $\tau \geq R^{d-1}$.) Hence (8.5) says that

$$(8.9) \quad \liminf_{R \rightarrow \infty} \int_0^\infty \tilde{N}(\tau) (-dA_R(\tau)) \geq C' := (1 - \kappa_{\mathcal{P}})C_{\mathcal{P}}v_d.$$

In view of (2.5) and Lemma 7 (with $\mathcal{D} = \mathcal{B}_1^d$), the statement of the present lemma is equivalent with $\liminf_{\tau \rightarrow \infty} \tau^{-d} \tilde{N}(\tau) \geq C'$. Assume that this is *false*. Then there is some $\eta > 0$ and a sequence $1 < \tau_1 < \tau_2 < \dots$ with $\tau_j \rightarrow \infty$ such that $\tilde{N}(\tau_j) < (1 - \eta)C'\tau_j^d$ for all j . Using the fact that $\tilde{N}(\tau)$ is an increasing function of τ we see that by shrinking $\eta > 0$ if necessary, we may actually assume that $\tilde{N}(\tau) < (1 - \eta)C'\tau^d$ for all $\tau \in [(1 - \eta)\tau_j, \tau_j]$ and all j . By Lemma 7 and (2.5) we have $\limsup_{\tau \rightarrow \infty} \tau^{-d} \tilde{N}(\tau) \leq C'$; thus for any given $\varepsilon > 0$ there is some $\tau_0 > 0$ such that $\tilde{N}(\tau) \leq (1 + \varepsilon)C'\tau^d$ for all $\tau \geq \tau_0$. Now for any j with $(1 - \eta)\tau_j > \tau_0$, and any $R > \tau_j^{1/(d-1)}$:

$$(8.10) \quad \int_0^\infty \tilde{N}(\tau) (-dA_R(\tau)) \leq \int_0^{\tau_0} \tilde{N}(\tau) (-dA_R(\tau)) + (1 + \varepsilon)C' \int_{\tau_0}^{R^{d-1}} \tau^d (-dA_R(\tau)) \\ - (\varepsilon + \eta)C' \int_{(1-\eta)\tau_j}^{\tau_j} \tau^d (-dA_R(\tau)).$$

Here the sum of the first two terms tends to $(1 + \varepsilon)C'$ as $R \rightarrow \infty$, as in [10, (5.11)-(5.13)]. Furthermore, if we choose $R = (2\tau_j)^{1/(d-1)}$ and let $j \rightarrow \infty$ then

$$(8.11) \quad \int_{(1-\eta)\tau_j}^{\tau_j} \tau^d (-dA_R(\tau)) = \frac{d}{\omega(S_1^{d-1})} \text{vol}\left(\mathcal{B}_1^d \Phi^{-\log R} \cap \mathcal{B}_{\frac{1}{2}R^{d-1}}^d \setminus \mathcal{B}_{\frac{1}{2}(1-\eta)R^{d-1}}^d\right) \\ \rightarrow \frac{2v_{d-1}}{v_d} \int_{(1-\eta)/2}^{1/2} (1-x^2)^{(d-1)/2} dx.$$

Hence we conclude that there is a constant $c(\eta) > 0$, independent of ε , such that

$$(8.12) \quad \liminf_{R \rightarrow \infty} \int_0^\infty \tilde{N}(\tau) (-dA_R(\tau)) \leq (1 + \varepsilon - c(\eta))C'.$$

Letting now $\varepsilon \rightarrow 0$ we run into a contradiction against (8.9). This concludes the proof of the lemma. \square

9. PROOF OF THEOREM 1

Combining Lemma 7 and Lemma 11 we can now complete the proof of Theorem 1. First let $\mathfrak{U}, \mathcal{D}$ be as in Lemma 7. Then by Lemma 7 applied to $S_1^{d-1} \setminus \mathfrak{U}$,

$$(9.1) \quad \liminf_{T \rightarrow \infty} \frac{\#(\hat{\mathcal{P}} \cap \mathcal{B}_T^d \setminus T\mathcal{D})}{T^d} \geq \kappa_{\mathcal{P}} C_{\mathcal{P}} (v_d - \text{vol}(\mathcal{D})).$$

Combining this with Lemma 11 we get

$$(9.2) \quad \limsup_{T \rightarrow \infty} \frac{\#(\hat{\mathcal{P}} \cap T\mathcal{D})}{T^d} = \limsup_{T \rightarrow \infty} \left(\frac{\#(\hat{\mathcal{P}} \cap \mathcal{B}_T^d)}{T^d} - \frac{\#(\hat{\mathcal{P}} \cap \mathcal{B}_T^d \setminus T\mathcal{D})}{T^d} \right) \leq \kappa_{\mathcal{P}} C_{\mathcal{P}} \text{vol}(\mathcal{D}).$$

Combining this with Lemma 7 (applied to \mathfrak{U} itself) we conclude

$$(9.3) \quad \lim_{T \rightarrow \infty} \frac{\#(\hat{\mathcal{P}} \cap T\mathcal{D})}{T^d} = \kappa_{\mathcal{P}} C_{\mathcal{P}} \text{vol}(\mathcal{D}).$$

By a scaling and subtraction argument it follows that (9.3) is true more generally for any $\mathcal{D} \in \mathcal{F}$, where \mathcal{F} is the family of sets of the form $\mathcal{D} = \{\mathbf{v} \in \mathbb{R}^d : r_1 \leq \|\mathbf{v}\| < r_2, \mathbf{v} \in \|\mathbf{v}\|\mathfrak{U}\}$, for any $0 \leq r_1 < r_2$ and any $\mathfrak{U} \subset S_1^{d-1}$ with $\omega(\partial\mathfrak{U}) = 0$.

Now let \mathcal{D} be an arbitrary subset of \mathbb{R}^d with boundary of measure zero. Note that the validity of (9.3) does not change if we replace \mathcal{D} by $\mathcal{D} \cup \{\mathbf{0}\}$ or by $\mathcal{D} \setminus \{\mathbf{0}\}$. The proof of Theorem 6 is now completed by approximating $\mathcal{D} \cup \{\mathbf{0}\}$ from above and $\mathcal{D} \setminus \{\mathbf{0}\}$ from below by finite unions of sets in \mathcal{F} .

10. PROOF OF THEOREM 2

Recall that (1.7) was proved in [10, Thm. A.1] and we have proved (1.9) and (1.10) in Section 5. Also the continuity of $E(r, \sigma, \mathcal{P})$ and $E(r, \sigma, \hat{\mathcal{P}})$ with respect to σ is immediate from (3.5), (3.6) combined with Theorem 6. Hence it remains to prove (1.8).

Thus let λ be a Borel probability measure on S_1^{d-1} which is absolutely continuous with respect to ω , and let $\sigma > 0$ and $r \in \mathbb{Z}_{\geq 0}$. Let us fix, once and for all, a map $K : S_1^{d-1} \rightarrow \text{SO}(d)$ such that $\mathbf{v}K(\mathbf{v}) = \mathbf{e}_1 = (1, 0, \dots, 0)$ for all $\mathbf{v} \in S_1^{d-1}$; we assume that K is smooth when restricted to S_1^{d-1} minus one point, cf. [9, Footnote 3, p. 1968]. Recall the definitions of $\mathfrak{C}(\sigma)$ and Φ^t in (3.7) and (8.3).

On verifies that if $\sigma', \sigma'', \alpha$ are any fixed numbers satisfying $0 < \sigma' < \sigma < \sigma''$ and $\sigma'/\sigma < \alpha < 1$, then for any $\mathbf{v} \in S_1^{d-1}$ and all sufficiently large T , the set of $\mathbf{y} \in \mathcal{B}_T^d \setminus \{\mathbf{0}\}$ satisfying $\|\mathbf{y}\|^{-1}\mathbf{y} \in \mathfrak{D}_T(\kappa_{\mathcal{P}}^{-1}\sigma, \mathbf{v})$ is contained in $\mathfrak{C}(\kappa_{\mathcal{P}}^{-1}\sigma'')\Phi^{-(\log T)/(d-1)}K(\mathbf{v})^{-1}$, and contains

$\mathfrak{C}(\kappa_{\mathcal{P}}^{-1}\sigma')\Phi^{-(\log(\alpha T))/(d-1)}K(\mathbf{v})^{-1}$. It follows that

$$(10.1) \quad \begin{aligned} & \lambda(\{\mathbf{v} \in S_1^{d-1} : \#(\widehat{\mathcal{P}} \cap \mathfrak{C}(\kappa_{\mathcal{P}}^{-1}\sigma'')\Phi^{-(\log T)/(d-1)}K(\mathbf{v})^{-1}) \leq r\}) \\ & \leq \lambda(\{\mathbf{v} \in S_1^{d-1} : \mathcal{N}_T(\sigma, \mathbf{v}, \widehat{\mathcal{P}}) \leq r\}) \\ & \leq \lambda(\{\mathbf{v} \in S_1^{d-1} : \#(\widehat{\mathcal{P}} \cap \mathfrak{C}(\kappa_{\mathcal{P}}^{-1}\sigma')\Phi^{-(\log(\alpha T))/(d-1)}K(\mathbf{v})^{-1}) \leq r\}). \end{aligned}$$

Recalling the definition of $\mathcal{P} = \mathcal{P}(\mathcal{W}, \mathbb{Z}^n g)$ we see that $\widehat{\mathcal{P}}A = \widehat{\mathcal{P}}(\mathcal{W}, \mathbb{Z}^n \varphi_g(A)g)$ for any $A \in \text{SL}(d, \mathbb{R})$. Hence if we define

$$(10.2) \quad \mathcal{E}(\sigma, r) = \{x \in X : \#(\widehat{\mathcal{P}}^x \cap \mathfrak{C}(\kappa_{\mathcal{P}}^{-1}\sigma)) \leq r\},$$

then the left hand side in (10.1) equals

$$(10.3) \quad \lambda(\{\mathbf{v} \in S_1^{d-1} : \Gamma\varphi_g(K(\mathbf{v})\Phi^{(\log T)/(d-1)}) \in \mathcal{E}(\sigma'', r)\})$$

Hence by [10, Thm. 4.1] and the Portmanteau theorem:

$$(10.4) \quad \liminf_{T \rightarrow \infty} \lambda(\{\mathbf{v} \in S_1^{d-1} : \mathcal{N}_T(\sigma, \mathbf{v}, \widehat{\mathcal{P}}) \leq r\}) \geq \mu(\mathcal{E}(\sigma'', r)^\circ) = \mu(\mathcal{E}(\sigma'', r)).$$

Here the last equality is proved by using Lemma 10 with $U = \mathfrak{C}(\kappa_{\mathcal{P}}^{-1}\sigma'')$, and noticing that Theorem 6 implies that $\widehat{\mathcal{P}}^x \cap \partial U = \emptyset$ for μ -almost all $x \in X$. Similarly, using the right relation in (10.1), we obtain

$$(10.5) \quad \limsup_{T \rightarrow \infty} \lambda(\{\mathbf{v} \in S_1^{d-1} : \mathcal{N}_T(\sigma, \mathbf{v}, \widehat{\mathcal{P}}) \leq r\}) \leq \mu(\overline{\mathcal{E}(\sigma', r)}) = \mu(\mathcal{E}(\sigma', r)).$$

Note that $\mathcal{E}(\sigma'', r) \subset \mathcal{E}(\sigma, r) \subset \mathcal{E}(\sigma', r)$, since $\mathfrak{C}(\kappa_{\mathcal{P}}^{-1}\sigma'') \supset \mathfrak{C}(\kappa_{\mathcal{P}}^{-1}\sigma) \supset \mathfrak{C}(\kappa_{\mathcal{P}}^{-1}\sigma')$. Also, if x lies in $\mathcal{E}(\sigma, r)$ but not in $\mathcal{E}(\sigma'', r)$, then $\widehat{\mathcal{P}}^x$ must have some point in $\mathfrak{C}(\kappa_{\mathcal{P}}^{-1}\sigma'') \setminus \mathfrak{C}(\kappa_{\mathcal{P}}^{-1}\sigma)$, and so by Theorem 6,

$$(10.6) \quad \mu(\mathcal{E}(\sigma, r)) - \mu(\mathcal{E}(\sigma'', r)) \leq \kappa_{\mathcal{P}} C_{\mathcal{P}} \text{vol}(\mathfrak{C}(\kappa_{\mathcal{P}}^{-1}\sigma'') \setminus \mathfrak{C}(\kappa_{\mathcal{P}}^{-1}\sigma)).$$

Similarly

$$(10.7) \quad \mu(\mathcal{E}(\sigma', r)) - \mu(\mathcal{E}(\sigma, r)) \leq \kappa_{\mathcal{P}} C_{\mathcal{P}} \text{vol}(\mathfrak{C}(\kappa_{\mathcal{P}}^{-1}\sigma) \setminus \mathfrak{C}(\kappa_{\mathcal{P}}^{-1}\sigma')).$$

Now by taking σ', σ'' sufficiently near σ , the right hand sides of (10.6) and (10.7) can be made as small as we like. Hence from (10.4) and (10.5) we obtain in fact

$$(10.8) \quad \lim_{T \rightarrow \infty} \lambda(\{\mathbf{v} \in S_1^{d-1} : \mathcal{N}_T(\sigma, \mathbf{v}, \widehat{\mathcal{P}}) \leq r\}) = \mu(\mathcal{E}(\sigma, r)) = \mu(\{x \in X : \#(\widehat{\mathcal{P}}^x \cap \mathfrak{C}(\kappa_{\mathcal{P}}^{-1}\sigma)) \leq r\}).$$

Note here that the right hand side is the same as $\sum_{r'=0}^r E(r, \sigma, \widehat{\mathcal{P}})$; cf. (3.6). Hence since (10.8) has been proved for arbitrary $r \geq 0$, also (1.8) holds for arbitrary $r \geq 0$, and we are done.

11. PROOF OF COROLLARY 3

It follows from Theorem 2 and a general statistical argument (cf. e.g. [8]) that if we define $F(0) = 0$ and

$$(11.1) \quad F(s) = -\frac{d}{ds}E(0, s, \widehat{\mathcal{P}}),$$

then the limit relation (1.14) holds at each point $s \geq 0$ where $F(s)$ is defined. In fact the function $s \mapsto E(0, s, \widehat{\mathcal{P}})$ is convex; hence $F(s)$ exists for all $s > 0$ except at most a countable number of points, and is continuous at each point where it exists. Also $F(s)$ is decreasing, and satisfies $\lim_{s \rightarrow 0^+} F(s) = 1 = F(0)$ (cf. (1.10)) and $\lim_{s \rightarrow \infty} F(s) = 0$. Note also that (1.15) is an immediate consequence of (1.14), the definition of $\widehat{\xi}_{T,j}$ and the fact that $N(T) \sim \kappa_{\mathcal{P}}^{-1}\widehat{N}(T)$ as $T \rightarrow \infty$ (cf. Theorem 1 and (1.6)).

It now only remains to prove that $F(s)$ is continuous, or equivalently that the derivative in (11.1) exists for every $s > 0$. Assume the contrary, and let $s_0 > 0$ be a point where the derivative does *not* exist. By convexity, both the left and right derivative exist at s_0 ; thus

$$(11.2) \quad -\lim_{s \rightarrow s_0^-} \frac{E(0, s_0, \widehat{\mathcal{P}}) - E(0, s, \widehat{\mathcal{P}})}{s_0 - s} > -\lim_{s \rightarrow s_0^+} \frac{E(0, s, \widehat{\mathcal{P}}) - E(0, s_0, \widehat{\mathcal{P}})}{s - s_0} \geq 0.$$

In dimension $d = 2$, using the fact that the point process $x \mapsto \widehat{\mathcal{P}}^x$ is invariant under $\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$ for any $r \in \mathbb{R}$, it follows that the formula (3.5) holds with $\mathfrak{C}(\sigma)$ replaced by $\mathfrak{C}(a, a + \sigma)$ for any $a \in \mathbb{R}$, where

$$\mathfrak{C}(a_1, a_2) = \left\{ \mathbf{y} = (y_1, y_2) \in \mathbb{R}^2 : 0 < y_1 < 1, \frac{2}{\kappa_{\mathcal{P}} C_{\mathcal{P}}} a_1 y_1 < y_2 < \frac{2}{\kappa_{\mathcal{P}} C_{\mathcal{P}}} a_2 y_1 \right\}$$

In particular, for any $0 < s < s'$ and $a \in \mathbb{R}$,

$$(11.3) \quad E(0, s, \widehat{\mathcal{P}}) - E(0, s', \widehat{\mathcal{P}}) = \mu(\{x \in X : \widehat{\mathcal{P}}^x \cap \mathfrak{C}(a, a + s) = \emptyset, \widehat{\mathcal{P}}^x \cap \mathfrak{C}(a, a + s') \neq \emptyset\}).$$

For given $x \in X$, we order the numbers

$$\frac{\kappa_{\mathcal{P}} C_{\mathcal{P}}}{2} \cdot \frac{y_2}{y_1} \quad \text{for } \mathbf{y} = (y_1, y_2) \in \widehat{\mathcal{P}}^x \cap ((0, 1) \times \mathbb{R}_{>0})$$

as $0 < \lambda_{x,1} < \lambda_{x,2} < \dots$. We also set $\lambda_{x,0} = 0$. Taking $s' = s_0 > s$ in (11.3), integrating over $a \in (0, a_0)$ for some fixed $a_0 > 0$, and using Fubini's Theorem, we obtain

$$a_0(E(0, s, \widehat{\mathcal{P}}) - E(0, s_0, \widehat{\mathcal{P}})) \leq \int_X (s_0 - s) \# \{j \geq 0 : \lambda_{x,j+1} - \lambda_{x,j} > s, \lambda_{x,j+1} < a_0 + s_0\} d\mu(x).$$

Hence

$$(11.4) \quad -a_0 \lim_{s \rightarrow s_0^-} \frac{E(0, s_0, \widehat{\mathcal{P}}) - E(0, s, \widehat{\mathcal{P}})}{s_0 - s} \leq \int_X \# \{j \geq 0 : \lambda_{x,j+1} - \lambda_{x,j} \geq s_0, \lambda_{x,j+1} < a_0 + s_0\} d\mu(x).$$

Similarly, replacing s by s_0 and s' by s in (11.3), we obtain

$$(11.5) \quad -a_0 \lim_{s \rightarrow s_0^+} \frac{E(0, s, \widehat{\mathcal{P}}) - E(0, s_0, \widehat{\mathcal{P}})}{s - s_0} \geq \int_X \# \{j \geq 0 : \lambda_{x,j+1} - \lambda_{x,j} > s_0, \lambda_{x,j+1} < a_0 + s_0\} d\mu(x).$$

It follows from (11.2), (11.4) and (11.5) that there is a set $A \subset X$ with $\mu(A) > 0$ such that for every $x \in A$, there is some $j \geq 0$ such that $\lambda_{x,j+1} - \lambda_{x,j} = s_0$ and $\lambda_{x,j} < a_0$. Note that $\lambda_{x,1} \neq s_0$ for μ -almost all $x \in X$, by Theorem 6 applied with f as the characteristic function of the line $y_2 = s_0 \frac{2}{\kappa_{\mathcal{P}} C_{\mathcal{P}}} y_1$ in \mathbb{R}^2 . Hence after removing a null set from A , we have for each $x \in A$ that $\widehat{\mathcal{P}}^x$ contains a pair of points $\mathbf{y} = (y_1, y_2)$ and $\mathbf{y}' = (y'_1, y'_2)$ satisfying

$$0 < y_1, y'_1 < 1, \quad \frac{y'_2}{y'_1} - \frac{y_2}{y_1} = \frac{2}{\kappa_{\mathcal{P}} C_{\mathcal{P}}} s_0, \quad 0 < \frac{y_2}{y_1} < \frac{2}{\kappa_{\mathcal{P}} C_{\mathcal{P}}} a_0.$$

However this is easily seen to violate the $\mathrm{SL}(2, \mathbb{R})$ -invariance of the point process $x \mapsto \widehat{\mathcal{P}}^x$. For example, for each $\frac{1}{2} \leq \lambda \leq 1$, because of the invariance under $\begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & 1/\sqrt{\lambda} \end{pmatrix}$, there is a subset $A_\lambda \subset X$ with $\mu(A_\lambda) = \mu(A) > 0$ such that for each $x \in A_\lambda$, $\widehat{\mathcal{P}}^x$ contains a pair of points $\mathbf{y} = (y_1, y_2)$ and $\mathbf{y}' = (y'_1, y'_2)$ satisfying

$$0 < y_1, y'_1 < \sqrt{\lambda}, \quad \frac{y'_2}{y'_1} - \frac{y_2}{y_1} = \frac{2}{\kappa_{\mathcal{P}} C_{\mathcal{P}}} \frac{s_0}{\lambda}, \quad 0 < \frac{y_2}{y_1} < \frac{2}{\kappa_{\mathcal{P}} C_{\mathcal{P}}} \frac{a_0}{\lambda}.$$

Let R be the rectangle $(0, 1) \times (0, \frac{4}{\kappa_{\mathcal{P}} C_{\mathcal{P}}}(a_0 + s_0))$ in \mathbb{R}^2 . By taking N sufficiently large we can ensure that the set $X_{R,N} := \{x \in X : \#(\widehat{\mathcal{P}}^x \cap R) \leq N\}$ has measure $\mu(X_{R,N}) \geq 1 - \frac{1}{2}\mu(A)$. It follows that $\mu(A_\lambda \cap X_{R,N}) \geq \frac{1}{2}\mu(A)$ for each $\frac{1}{2} \leq \lambda \leq 1$, and so if Λ is any infinite subset

of $[\frac{1}{2}, 1]$ then the integral $\int_{X_{R,N}} \sum_{\lambda \in \Lambda} I(x \in A_\lambda) d\mu(x)$ is infinite. On the other hand the definition of $X_{R,N}$ implies that $\sum_{\lambda \in \Lambda} I(x \in A_\lambda) \leq \binom{N}{2}$ for each $x \in X_{R,N}$.

We have thus reached a contradiction, and we conclude that (11.2) cannot hold, i.e. $F(s)$ is continuous for all $s \geq 0$.

12. VANISHING NEAR ZERO OF THE GAP DISTRIBUTION

The gap distribution obtained in Corollary 3 may sometimes vanish near zero. This phenomenon was noted numerically in [2] in several examples. In the case when \mathcal{P} is a *lattice*, this vanishing is well understood; cf. [3], [9]. We recall that in this case the limit distribution, and hence the gap size, is independent of the choice of lattice.

Let $\mathcal{P} = \mathcal{P}(\mathcal{W}, \mathcal{L})$ be a regular cut-and-project set. We define $m_{\widehat{\mathcal{P}}} \geq 0$ to be the supremum of all $\sigma \geq 0$ with the property that $\#(\widehat{\mathcal{P}}^x \cap \mathfrak{C}(\kappa_{\mathcal{P}}^{-1}\sigma)) \leq 1$ for (μ) -almost all $x \in X$. Then the computation in (5.3) (together with (5.2)) shows that

$$(12.1) \quad E(0, \sigma, \widehat{\mathcal{P}}) \quad \begin{cases} = 1 - \sigma & \text{when } 0 \leq \sigma \leq m_{\widehat{\mathcal{P}}} \\ > 1 - \sigma & \text{when } \sigma > m_{\widehat{\mathcal{P}}}. \end{cases}$$

We note that if $d \geq 3$ then $m_{\widehat{\mathcal{P}}} = 0$, because of the $\mathrm{SL}(d, \mathbb{R})$ -invariance and the fact that $\mathrm{SL}(d, \mathbb{R})$ acts transitively on pairs of non-proportional vectors in $\mathbb{R}^d \setminus \{\mathbf{0}\}$ when $d \geq 3$.

Let us now assume $d = 2$. Note that by (12.1) and the discussion at the beginning of Sec. 11, the function F in Corollary 3 satisfies

$$F(s) \quad \begin{cases} = 1 & \text{if } 0 \leq s \leq m_{\widehat{\mathcal{P}}} \\ < 1 & \text{if } s > m_{\widehat{\mathcal{P}}}. \end{cases}$$

In other words, $m_{\widehat{\mathcal{P}}}$ is the largest number with the property that the limiting gap distribution obtained in Corollary 3 is supported on the interval $[m_{\widehat{\mathcal{P}}}, \infty)$. In particular, the support of the limiting gap distribution is separated from 0 if and only if $m_{\widehat{\mathcal{P}}} > 0$.

Let us also note that if $d = 2$, $m \geq 1$, and \mathcal{L} is a “generic” lattice or affine lattice, so that either $H_g = \mathrm{SL}(n, \mathbb{R})$ or $H_g = G = \mathrm{ASL}(n, \mathbb{R})$, then we have $m_{\widehat{\mathcal{P}}} = 0$, again using the transitivity of the action of $\mathrm{SL}(n, \mathbb{R})$ on pairs of non-proportional vectors in $\mathbb{R}^n \setminus \{\mathbf{0}\}$ for $n \geq 3$.

On the other hand, we will now recall (for general d) a standard construction of cut-and-project sets using the geometric representation of algebraic numbers [4], which can be used to produce several of the most well-known quasicrystals, cf. [1, 11, 12, 13, 14, 16]. We will see that in special cases with $d = 2$, this construction leads to quasicrystals for which $m_{\widehat{\mathcal{P}}} > 0$.

We follow [10, Sec. 2.2]. Let K be a totally real number field of degree $N \geq 2$ over \mathbb{Q} , let \mathcal{O}_K be its subring of algebraic integers, and let π_1, \dots, π_N be the distinct embeddings of K into \mathbb{R} . We will always view K as a subset of \mathbb{R} via π_1 ; in other words we agree that π_1 is the identity map. Fix $d \geq 1$ and set $n = dN$. By abuse of notation we write π_j also for the coordinate-wise embedding of K^d into \mathbb{R}^d , and for the entry-wise embedding of $M_d(K)$ (the algebra of $d \times d$ matrices with entries in K) into $M_d(\mathbb{R})$. Let \mathcal{L} be the lattice in $\mathbb{R}^n = (\mathbb{R}^d)^N$ given by

$$(12.2) \quad \mathcal{L} = \mathcal{L}_K^d := \left\{ (\mathbf{x}, \pi_2(\mathbf{x}), \dots, \pi_N(\mathbf{x})) : \mathbf{x} \in \mathcal{O}_K^d \right\}.$$

As usual we set $m = n - d = (N - 1)d$, let π and π_{int} be the projections of $\mathbb{R}^n = (\mathbb{R}^d)^N = \mathbb{R}^d \times \mathbb{R}^m$ onto the first d and last m coordinates. It follows from [23, Cor. 2 in Ch. IV-2] that $\pi_{\mathrm{int}}(\mathcal{L})$ is dense in \mathbb{R}^m , i.e. we have $\mathcal{A} = \mathbb{R}^m$ and $\mathcal{V} = \mathbb{R}^n$ in the present situation. Hence the window \mathcal{W} should be taken as a subset of \mathbb{R}^m , and we consider the cut-and-project set $\mathcal{P}(\mathcal{W}, \mathcal{L}) \subset \mathbb{R}^d$.

Choose $\delta > 0$ and $g \in \mathrm{SL}(n, \mathbb{R})$ such that

$$(12.3) \quad \mathcal{L} = \delta^{1/n} \mathbb{Z}^n g.$$

In fact

$$(12.4) \quad \delta = |D_K|^{d/2},$$

where D_K is the discriminant of K ; cf., e.g., [7, Ch. V.2, Lemma 2]. As proved in [10, Sec. 2.2.1], in this situation we have

$$(12.5) \quad H_g = g \mathrm{SL}(d, \mathbb{R})^N g^{-1};$$

where $\mathrm{SL}(d, \mathbb{R})^N$ is embedded as a subgroup of $G = \mathrm{ASL}(n, \mathbb{R})$ through

$$(12.6) \quad (A_1, \dots, A_N) \mapsto (\mathrm{diag}[A_1, \dots, A_N], \mathbf{0}),$$

where $\mathrm{diag}[A_1, \dots, A_N]$ is the block matrix whose diagonal blocks are A_1, \dots, A_N in this order, and all other blocks vanish.

Lemma 12. *Let $\mathcal{P} = \mathcal{P}(\mathcal{W}, \mathcal{L})$ be a regular cut-and-project set with \mathcal{L} as in (12.2), and with $d = N = 2$ (thus K is a real quadratic number field). Let $\varepsilon > 1$ be the fundamental unit of \mathcal{O}_K , and set $R = \sup\{\|\mathbf{w}\| : \mathbf{w} \in \mathcal{W}\}$. Then*

$$(12.7) \quad m_{\hat{\mathcal{P}}} \geq \frac{\kappa_{\mathcal{P}} C_{\mathcal{P}} \delta}{(\varepsilon^2 + \varepsilon^{-2})^2 R^2}.$$

Proof. Let $\sigma > 0$ and $x \in X$ be given and assume that $\#(\hat{\mathcal{P}}^x \cap \mathfrak{C}(\kappa_{\mathcal{P}}^{-1} \sigma)) \geq 2$. It suffices to prove that we must then have $\sigma \geq \frac{\kappa_{\mathcal{P}} C_{\mathcal{P}} \delta}{(\varepsilon^2 + \varepsilon^{-2})^2 R^2}$. The area of $\mathfrak{C}(\kappa_{\mathcal{P}}^{-1} \sigma)$ equals r^2 where $r := \sqrt{\frac{\sigma}{\kappa_{\mathcal{P}} C_{\mathcal{P}}}}$; hence there is some $A \in \mathrm{SL}(2, \mathbb{R})$ which maps $\mathfrak{C}(\kappa_{\mathcal{P}}^{-1} \sigma)$ to the open triangle $\mathfrak{C}_r := \{\mathbf{x} \in \mathbb{R}^2 : 0 < x_1 < r, |x_2| < x_1\}$. Take $(A_1, A_2) \in \mathrm{SL}(2, \mathbb{R})^2$ (embedded in G as in (12.6)) so that $x = \Gamma g(A_1, A_2) g^{-1}$. Set $\tilde{A} = (A_1 A, A_2)$; then $\mathcal{P}^x A = \mathcal{P}(\mathcal{W}, \mathcal{L} \tilde{A})$. We set $\gamma = \mathrm{diag}[\varepsilon^{-k}, \varepsilon^{-k}, \varepsilon^k, \varepsilon^k] \in \mathrm{SL}(4, \mathbb{R})$, where k is an integer which we will choose below. Then $\mathcal{L} \tilde{A} = \mathcal{L} \gamma \tilde{A} = \mathcal{L} \tilde{A} \gamma$, by (12.2) and since \tilde{A} is block diagonal. Hence

$$\mathcal{P}^x A = \mathcal{P}(\mathcal{W}, \mathcal{L} \tilde{A}) = \mathcal{P}(\mathcal{W}, \mathcal{L} \tilde{A} \gamma) = \varepsilon^{-k} \mathcal{P}(\varepsilon^{-k} \mathcal{W}, \mathcal{L} \tilde{A}).$$

Now $\#(\hat{\mathcal{P}}^x A \cap \mathfrak{C}_r) \geq 2$ and thus $\mathcal{L} \tilde{A}$ contains two points in $(\varepsilon^k \mathfrak{C}_r) \times (\varepsilon^{-k} \mathcal{W})$ which have non-proportional images under π (the projection onto the physical space \mathbb{R}^2). In other words, there exist $\mathbf{x}, \mathbf{x}' \in \mathcal{O}_K^2 \subset \mathbb{R}^2$ which are linearly independent over \mathbb{R} (thus also over K) such that $\mathbf{b}_1 = (\mathbf{x}, \overline{\mathbf{x}}) \tilde{A}$ and $\mathbf{b}_2 = (\mathbf{x}', \overline{\mathbf{x}}') \tilde{A}$ lie in $(\varepsilon^k \mathfrak{C}_r) \times (\varepsilon^{-k} \mathcal{W})$. Here we write $\mathbf{x} \mapsto \overline{\mathbf{x}}$ for the nontrivial automorphism of K . It follows that also $\mathbf{b}_3 = (\varepsilon \mathbf{x}, \overline{\varepsilon \mathbf{x}}) \tilde{A}$ and $\mathbf{b}_4 = (\varepsilon \mathbf{x}', \overline{\varepsilon \mathbf{x}'}) \tilde{A}$ lie in $(\varepsilon^{k+1} \mathfrak{C}_r) \times (\varepsilon^{-k-1} \mathcal{W})$. However the four vectors $(\mathbf{x}, \overline{\mathbf{x}})$, $(\mathbf{x}', \overline{\mathbf{x}'})$, $(\varepsilon \mathbf{x}, \overline{\varepsilon \mathbf{x}})$, $(\varepsilon \mathbf{x}', \overline{\varepsilon \mathbf{x}'})$ lie in \mathcal{L} and form a K -linear basis of K^4 . Hence $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4$ lie in $\mathcal{L} \tilde{A}$ and are linearly independent over \mathbb{R} . However $\|\mathbf{b}_j\| < r'$ for $j = 1, 2, 3, 4$, where

$$r' = \max \left(\sqrt{(\varepsilon^k r)^2 + (\varepsilon^{-k} R)^2}, \sqrt{(\varepsilon^{k+1} r)^2 + (\varepsilon^{-k-1} R)^2} \right),$$

and thus δ , the covolume of $\mathcal{L} \tilde{A}$, must be less than r'^4 . Now choose k so as to minimize r' . Then $r' \leq \sqrt{\varepsilon^2 + \varepsilon^{-2}} \sqrt{Rr}$, and combining this with $\delta < r'^4$ and $r = \sqrt{\frac{\sigma}{\kappa_{\mathcal{P}} C_{\mathcal{P}}}}$ we obtain $\sigma > \frac{\kappa_{\mathcal{P}} C_{\mathcal{P}} \delta}{(\varepsilon^2 + \varepsilon^{-2})^2 R^2}$, as desired. \square

Let us make some further observations in this vein. First, note the general relation

$$\mathcal{P}(\mathcal{W}, q^{-1} \mathcal{L}) = q^{-1} \mathcal{P}(q \mathcal{W}, \mathcal{L}), \quad \forall q > 0 \text{ (real)}.$$

Using this relation with q an appropriate positive integer, it is clear that if \mathcal{L} is any lattice in \mathbb{R}^n such that the cut-and-project set $\mathcal{P} = \mathcal{P}(\mathcal{W}, \mathcal{L})$ satisfies $m_{\hat{\mathcal{P}}} > 0$ for every admissible window set \mathcal{W} (for example this holds when \mathcal{L} is as in Lemma 12), then $m_{\hat{\mathcal{P}}} > 0$ also holds for any cut-and-project set obtained from $\mathcal{P}(\mathcal{W}, \mathcal{L})$ by the “union of rational translates” construction in [10, Sec. 2.3.1]. Furthermore, the property of having $m_{\hat{\mathcal{P}}} > 0$ is also, obviously, preserved

under “passing to a sublattice” as in [10, Sec. 2.4]. In particular, by [10, Sec. 2.5] and Remark 12.1 below, we have $m_{\widehat{\mathcal{P}}} > 0$ for any \mathcal{P} associated with a rhombic Penrose tiling.

Remark 12.1. If we wish to reproduce the vertex set of an *arbitrary* rhombic Penrose tiling (RPT) as a cut-and-project set within the present framework, we also need to consider the case of so-called *singular* vectors γ , as explained by de Bruijn [5] (we use the same notation as in [10, Sec. 2.5]). In this case there are either 2 or 10 distinct RPT’s associated to γ , and by [5, Sec. 12] (carried over to our notation), the vertex set of any of these can be expressed as

$$\{\pi(\mathbf{y}) : \mathbf{y} \in \mathcal{L}, [\exists M > 0 : \forall m > M : \pi_{\text{int}}(\mathbf{y}) \in \mathcal{W}(\gamma^{(m)})]\},$$

where $\gamma^{(m)}$ is an appropriate sequence of regular vectors tending to γ as $m \rightarrow \infty$, and we write $\mathcal{W} = \mathcal{W}(\gamma)$ for the open window set defined in [10, (2.25)]. In other words, the vertex set of the RPT equals $\mathcal{P}(\widetilde{\mathcal{W}}, \mathcal{L})$, where $\widetilde{\mathcal{W}} := \{\mathbf{v} \in \mathcal{A} : [\exists M > 0 : \forall m > M : \mathbf{v} \in \mathcal{W}(\gamma^{(m)})]\}$. Note that $\widetilde{\mathcal{W}}$ is the union of the open set $\mathcal{W}(\gamma)$ and part of its boundary. In particular $\partial\widetilde{\mathcal{W}} = \partial\mathcal{W}(\gamma)$ has measure zero with respect to $\mu_{\mathcal{A}}$. Hence the vertex set is again a regular cut-and-project set, and the previous discussion leading to $m_{\widehat{\mathcal{P}}} > 0$ applies.

Remark 12.2. We do not expect the lower bound in Lemma 12 to be sharp, and the argument which we gave regarding the construction in [10, Sec. 2.3.1] certainly does not lead to a sharp bound. It would be interesting to try to determine the *exact* value of $m_{\widehat{\mathcal{P}}}$ for the Penrose tiling, and also for some of the cases discussed in [2].

It is interesting to note that for a large class of regular cut-and-project sets with $m_{\widehat{\mathcal{P}}} > 0$, a corresponding lower bound on the gap length is present in the set of directions (1.13) not only in the limit $T \rightarrow \infty$, but for *any fixed* T :

Lemma 13. *Let $\mathcal{P} = \mathcal{P}(\mathcal{W}, \mathcal{L})$ be a regular cut-and-project set in dimension $d = 2$ such that either $\mathbf{0} \notin \mathcal{P}$ or $\mathbf{0} \in \mathcal{P}^x$ for all $x \in X$, and furthermore $\pi_{\text{int}}(\mathbf{y}) \notin \partial\mathcal{W}$ for all $\mathbf{y} \in \mathcal{L}$ (viz., there are no “singular vertices”; cf. [2, p. 6]). Then for any non-proportional vectors $\mathbf{p}_1, \mathbf{p}_2 \in \widehat{\mathcal{P}}$, the triangle with vertices $\mathbf{0}, \mathbf{p}_1, \mathbf{p}_2$ has area $\geq (\kappa_{\mathcal{P}} C_{\mathcal{P}})^{-1} m_{\widehat{\mathcal{P}}}$. In particular, for any $T > 0$ and $1 \leq j \leq \widehat{N}(T)$ we have $\widehat{\xi}_{T,j} - \widehat{\xi}_{T,j-1} \geq \min(\frac{1}{2}, (\pi \kappa_{\mathcal{P}} C_{\mathcal{P}})^{-1} m_{\widehat{\mathcal{P}}} T^{-2})$.*

(Using the last bound of Lemma 13 together with $\widehat{N}(T) \sim \pi \kappa_{\mathcal{P}} C_{\mathcal{P}} T^2$ as $T \rightarrow \infty$ in the limit relation (1.14) in Corollary 3, we immediately recover the fact that $F(s) = 1$ for $0 \leq s \leq m_{\widehat{\mathcal{P}}}$. We also remark that the condition $\mathbf{0} \in \mathcal{P}^x$ for all $x \in X$ is fulfilled whenever $\mathbf{0} \in \mathcal{W}$ and \mathcal{L} is a lattice, since then $H_g \subset \text{SL}(n, \mathbb{R})$.)

Proof. Assume that $\mathbf{p}_1, \mathbf{p}_2 \in \widehat{\mathcal{P}}$ are non-proportional vectors and that the triangle $\Delta(\mathbf{0}, \mathbf{p}_1, \mathbf{p}_2)$ has area less than $(\kappa_{\mathcal{P}} C_{\mathcal{P}})^{-1} m_{\widehat{\mathcal{P}}}$. Note that for any $\mathbf{p}'_1, \mathbf{p}'_2 \in \mathbb{R}^2$ such that $\Delta(\mathbf{0}, \mathbf{p}'_1, \mathbf{p}'_2)$ has the same area and orientation as $\Delta(\mathbf{0}, \mathbf{p}_1, \mathbf{p}_2)$, there exists $A \in \text{SL}(2, \mathbb{R})$ with $\mathbf{p}'_1 = \mathbf{p}_1 A$ and $\mathbf{p}'_2 = \mathbf{p}_2 A$. In particular there are some $A \in \text{SL}(2, \mathbb{R})$ and $\sigma_0 \in (0, m_{\widehat{\mathcal{P}}})$ such that $\mathbf{p}_1 A, \mathbf{p}_2 A \in \mathfrak{C}(\kappa_{\mathcal{P}}^{-1} \sigma_0)$. Now there are $\mathbf{y}_1, \mathbf{y}_2 \in \mathcal{L}$ such that $\pi(\mathbf{y}_j) = \mathbf{p}_j$ and $\pi_{\text{int}}(\mathbf{y}_j) \in \mathcal{W}$ for $j = 1, 2$, and by assumption neither $\pi_{\text{int}}(\mathbf{y}_1)$ nor $\pi_{\text{int}}(\mathbf{y}_2)$ lie in $\partial\mathcal{W}$; hence $\mathbf{y}_j \begin{pmatrix} A & 0 \\ 0 & 1_m \end{pmatrix} \in \mathfrak{C}(\kappa_{\mathcal{P}}^{-1} \sigma_0) \times \mathcal{W}^\circ$ for $j = 1, 2$. It follows that $\#(\widehat{\mathcal{P}}^x \cap \mathfrak{C}(\kappa_{\mathcal{P}}^{-1} \sigma_0)) \geq 2$ for $x = \Gamma\varphi_g(A) \in X$. In fact, using our assumptions on \mathcal{P} and the fact that $\mathfrak{C}(\kappa_{\mathcal{P}}^{-1} \sigma_0) \times \mathcal{W}^\circ$ is open, we have $\#(\widehat{\mathcal{P}}^{x'} \cap \mathfrak{C}(\kappa_{\mathcal{P}}^{-1} \sigma_0)) \geq 2$ for all x' in some open neighbourhood of $x = \Gamma\varphi_g(A)$ (cf. the proof of Lemma 10). However this violates our definition of $m_{\widehat{\mathcal{P}}}$. We have thus proved the first part of the lemma.

To prove the second statement we merely have to note that $\widehat{\xi}_{T,j} - \widehat{\xi}_{T,j-1} = (2\pi)^{-1} \varphi(\mathbf{p}_1, \mathbf{p}_2)$ for some $\mathbf{p}_1 \neq \mathbf{p}_2 \in \widehat{\mathcal{P}}_T$. If $\mathbf{p}_1, \mathbf{p}_2$ are not proportional then since $\Delta(\mathbf{0}, \mathbf{p}_1, \mathbf{p}_2)$ has area $\frac{1}{2} \|\mathbf{p}_1\| \|\mathbf{p}_2\| \sin \varphi(\mathbf{p}_1, \mathbf{p}_2) < \frac{1}{2} T^2 \sin \varphi(\mathbf{p}_1, \mathbf{p}_2)$, the first part of the lemma implies $\varphi(\mathbf{p}_1, \mathbf{p}_2) > \sin \varphi(\mathbf{p}_1, \mathbf{p}_2) > 2(\kappa_{\mathcal{P}} C_{\mathcal{P}})^{-1} m_{\widehat{\mathcal{P}}} T^{-2}$; on the other hand if $\mathbf{p}_1, \mathbf{p}_2$ are proportional then necessarily $\varphi(\mathbf{p}_1, \mathbf{p}_2) = \pi$. \square

APPENDIX: NON-SPHERICAL TRUNCATIONS

We have defined \mathcal{P}_T as the intersection of $\mathcal{P} \setminus \{\mathbf{0}\}$ and the ball \mathcal{B}_T^d . However, as we will now explain, the fact that λ is arbitrary in Theorem 2 allows us to obtain corresponding results with \mathcal{B}_T^d replaced by a more general expanding domain.

Throughout this section, let \mathcal{P} be an arbitrary point set with constant density $\theta(\mathcal{P})$, and let \mathcal{E} be a starshaped region in \mathbb{R}^d of the form

$$\mathcal{E} = \{r\mathbf{v} : \mathbf{v} \in S_1^{d-1}, 0 \leq r < \ell(\mathbf{v})\},$$

where ℓ is a continuous function from S_1^{d-1} to $\mathbb{R}_{>0}$. Set

$$(A.1) \quad \mathcal{N}_T(\sigma, \mathbf{v}, \mathcal{P}, \mathcal{E}) := \#\{\mathbf{y} \in \mathcal{P} \cap T\mathcal{E} \setminus \{\mathbf{0}\} : \|\mathbf{y}\|^{-1}\mathbf{y} \in \mathfrak{D}_T(\sigma, \mathbf{v})\}.$$

In particular, $\mathcal{N}_T(\sigma, \mathbf{v}, \mathcal{P}) = \mathcal{N}_T(\sigma, \mathbf{v}, \mathcal{P}, \mathcal{B}_1^d)$; cf. (1.4).

It is natural to rescale σ by a factor $\ell(\mathbf{v})^{-d}$ in (A.1), since we then recover the property that the expectation value is asymptotically constant and independent of the direction: For any probability measure λ on S_1^{d-1} with continuous density, we have

$$(A.2) \quad \lim_{T \rightarrow \infty} \int_{S_1^{d-1}} \mathcal{N}_T(\ell(\mathbf{v})^{-d}\sigma, \mathbf{v}, \mathcal{P}, \mathcal{E}) d\lambda(\mathbf{v}) = \sigma.$$

This generalizes (1.5), and again follows from (1.1). The proposition below covers both the rescaled and the original distribution.

Proposition 14. *Let $r \in \mathbb{Z}_{\geq 0}$. Assume that, for every $\sigma > 0$ and every Borel probability measure λ on S_1^{d-1} which is absolutely continuous with respect to ω , the limit*

$$(A.3) \quad \tilde{E}(r, \sigma, \mathcal{P}) := \lim_{T \rightarrow \infty} \lambda(\{\mathbf{v} \in S_1^{d-1} : \mathcal{N}_T(\sigma, \mathbf{v}, \mathcal{P}, \mathcal{B}_1^d) \leq r\})$$

exists and is continuous in σ and independent of λ . Then, for every σ and λ as above, we have

$$(A.4) \quad \lim_{T \rightarrow \infty} \lambda(\{\mathbf{v} \in S_1^{d-1} : \mathcal{N}_T(\ell(\mathbf{v})^{-d}\sigma, \mathbf{v}, \mathcal{P}, \mathcal{E}) \leq r\}) = \tilde{E}(r, \sigma, \mathcal{P})$$

and

$$(A.5) \quad \lim_{T \rightarrow \infty} \lambda(\{\mathbf{v} \in S_1^{d-1} : \mathcal{N}_T(\sigma, \mathbf{v}, \mathcal{P}, \mathcal{E}) \leq r\}) = \int_{S_1^{d-1}} \tilde{E}(r, \ell(\mathbf{v})^d\sigma, \mathcal{P}) d\lambda(\mathbf{v}).$$

In other words, the existence of a limit distribution of $\mathcal{N}_T(\sigma, \mathbf{v}, \mathcal{P}, \mathcal{B}_1^d)$ independent of λ implies the existence the limit distributions of both $\mathcal{N}_T(\sigma, \mathbf{v}, \mathcal{P}, \mathcal{E})$ and $\mathcal{N}_T(\ell(\mathbf{v})^{-d}\sigma, \mathbf{v}, \mathcal{P}, \mathcal{E})$, where the limit of the latter is in fact independent of \mathcal{E} !

Proof. This is a relatively standard approximation argument. We give the proof of (A.5); the proof of (A.4) is very similar.

Let r, σ, λ be given. For W an arbitrary measurable subset of S_1^{d-1} , let $\lambda|_W$ be the restriction of λ to W , and set $W_T := \cup_{\mathbf{v} \in W} \mathfrak{D}_T(\sigma, \mathbf{v})$ and $\ell_T^- = \inf\{\ell(\mathbf{v}) : \mathbf{v} \in W_T\}$. Then for any $0 < T_0 \leq T$ we have

$$\begin{aligned} \lambda|_W(\{\mathbf{v} \in S_1^{d-1} : \mathcal{N}_T(\sigma, \mathbf{v}, \mathcal{P}, \mathcal{E}) \leq r\}) &\leq \lambda|_W(\{\mathbf{v} \in S_1^{d-1} : \mathcal{N}_T(\sigma, \mathbf{v}, \mathcal{P}, \mathcal{B}_{\ell_T^-}^d) \leq r\}) \\ &= \lambda|_W(\{\mathbf{v} \in S_1^{d-1} : \mathcal{N}_{T\ell_T^-}((\ell_T^-)^d\sigma, \mathbf{v}, \mathcal{P}, \mathcal{B}_1^d) \leq r\}) \\ &\leq \lambda|_W(\{\mathbf{v} \in S_1^{d-1} : \mathcal{N}_{T\ell_T^-}((\ell_{T_0}^-)^d\sigma, \mathbf{v}, \mathcal{P}, \mathcal{B}_1^d) \leq r\}). \end{aligned}$$

If $\lambda(W) > 0$ then we get, letting $T \rightarrow \infty$ and using (A.3) with $\lambda(W)^{-1}\lambda|_W$ in place of λ :

$$\limsup_{T \rightarrow \infty} \lambda|_W(\{\mathbf{v} \in S_1^{d-1} : \mathcal{N}_T(\sigma, \mathbf{v}, \mathcal{P}, \mathcal{E}) \leq r\}) \leq \lambda(W) \tilde{E}(r, (\ell_{T_0}^-)^d\sigma, \mathcal{P}),$$

for any $T_0 > 0$. Letting here $T_0 \rightarrow \infty$, we conclude

$$(A.6) \quad \limsup_{T \rightarrow \infty} \lambda|_W(\{\mathbf{v} \in S_1^{d-1} : \mathcal{N}_T(\sigma, \mathbf{v}, \mathcal{P}, \mathcal{E}) \leq r\}) \leq \lambda(W) \tilde{E}(r, (\ell_W^-)^d\sigma, \mathcal{P}),$$

where $\ell_W^- = \inf_W \ell(\mathbf{v})$. Note that (A.6) also holds if $\lambda(W) = 0$, trivially.

Given any $\varepsilon > 0$, since $\tilde{E}(r, \sigma, \mathcal{P})$ is continuous in σ and $\ell(\mathbf{v})$ is uniformly continuous in \mathbf{v} , we can find a partition of S_1^{d-1} into measurable subsets W_1, \dots, W_m such that $|\tilde{E}(r, (\ell_{W_j}^-)^d \sigma, \mathcal{P}) - \tilde{E}(r, \ell(\mathbf{v})^d \sigma, \mathcal{P})| < \varepsilon$ for all $j \in \{1, \dots, m\}$ and all $\mathbf{v} \in W_j$. Using $\lambda = \sum_{j=1}^m \lambda(W_j) \lambda_{W_j}$, and applying (A.6) for each W_j , we get

$$\begin{aligned} \limsup_{T \rightarrow \infty} \lambda(\{\mathbf{v} \in S_1^{d-1} : \mathcal{N}_T(\sigma, \mathbf{v}, \mathcal{P}, \mathcal{E}) \leq r\}) &\leq \sum_{j=1}^m \lambda(W_j) \tilde{E}(r, (\ell_{W_j}^-)^d \sigma, \mathcal{P}) \\ &< \int_{S_1^{d-1}} \tilde{E}(r, \ell(\mathbf{v})^d \sigma, \mathcal{P}) d\lambda(\mathbf{v}) + \varepsilon. \end{aligned}$$

Similarly one proves a corresponding lower bound for the \liminf . Now (A.5) follows upon letting $\varepsilon \rightarrow 0$. \square

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